Time-frequency analysis

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Abstract:
Time-frequency analysis is
the part of Fourier analysis studying simultaneously
WHEN and HOW OFTEN
something happens in a signal.
With sharp time-frequency localizations we can
apply sharp time-frequency operations to signals, having
real-life applications in all fields of engineering and science.

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Idea of time-frequency analysis

Signals

$u, v : \mathbb{R} \to \mathbb{C}$

of finite energy: $u, v \in L^2(\mathbb{R})$.

Time-frequency transform

$Q(u, v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$,

time-frequency distribution

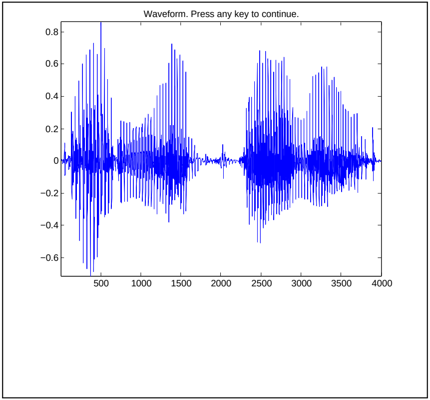
$Q(u, u) = Q[u] : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$.

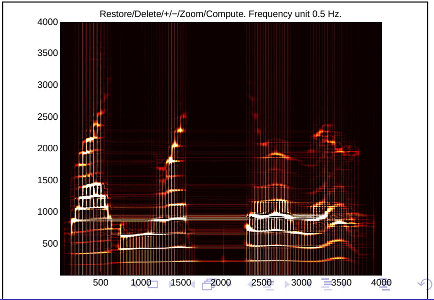
Time-frequency weight (symbol)

$\sigma : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$,

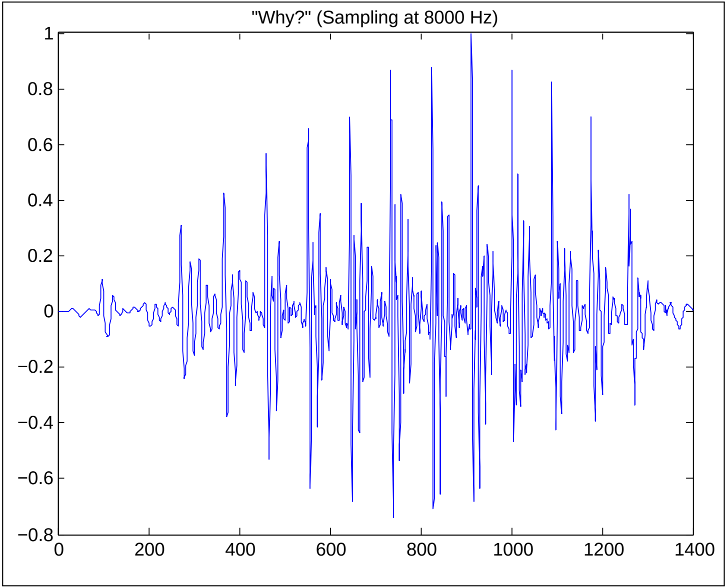
Q-quantization $\sigma \mapsto A_\sigma = A_{Q,\sigma}$:

$$\langle u, A_\sigma v \rangle_{L^2(\mathbb{R})} = \langle Q(u, v), \sigma \rangle_{L^2(\mathbb{R} \times \widehat{\mathbb{R}})}.$$

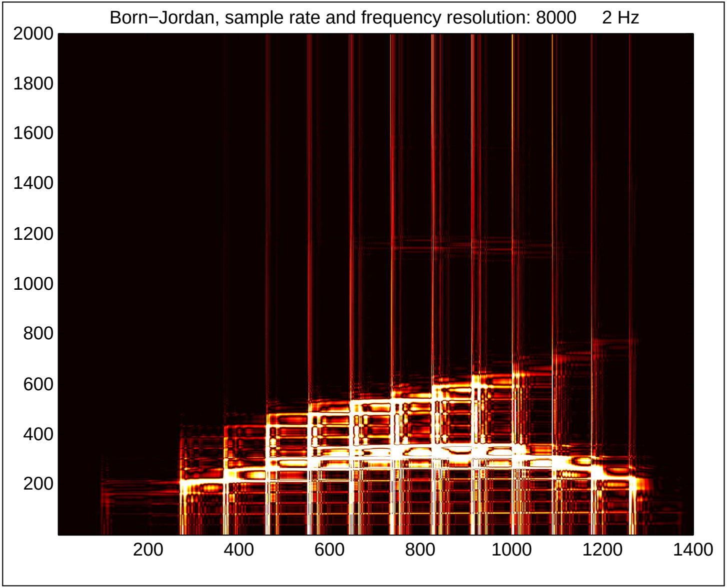




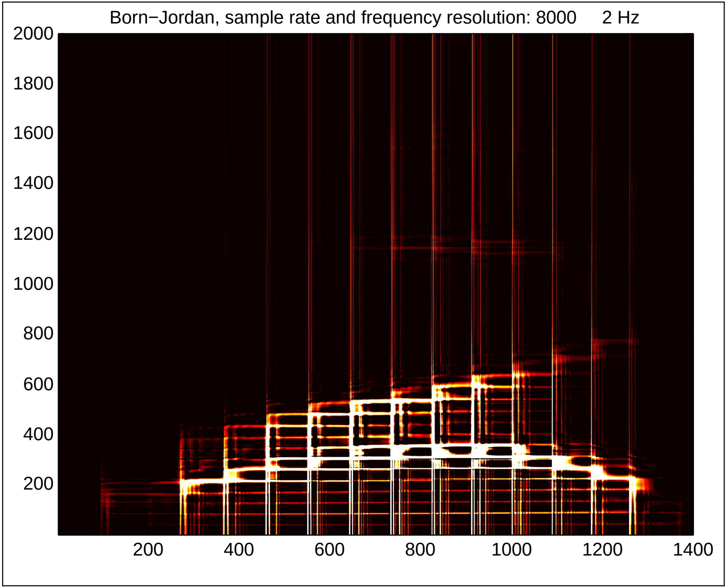
“Why?”


"Why?" (Sampling at 8000 Hz)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
200
400
600
800
1000
1200
1400

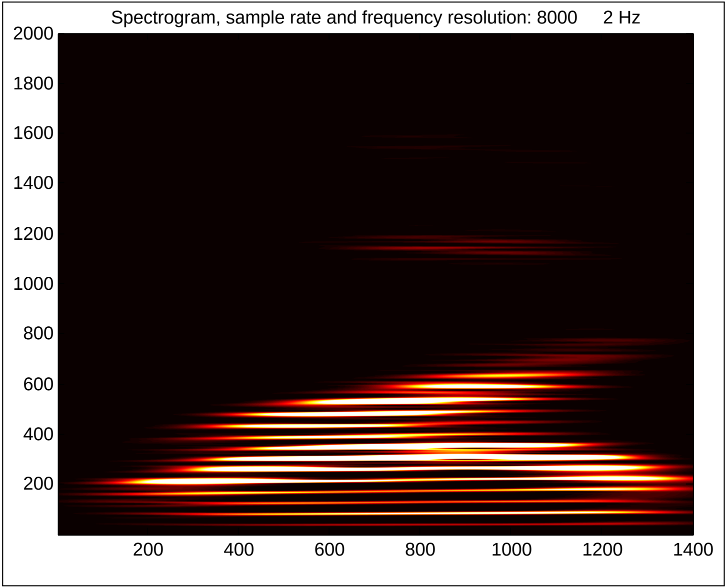
“Why?”: Absolute value of Born–Jordan distribution



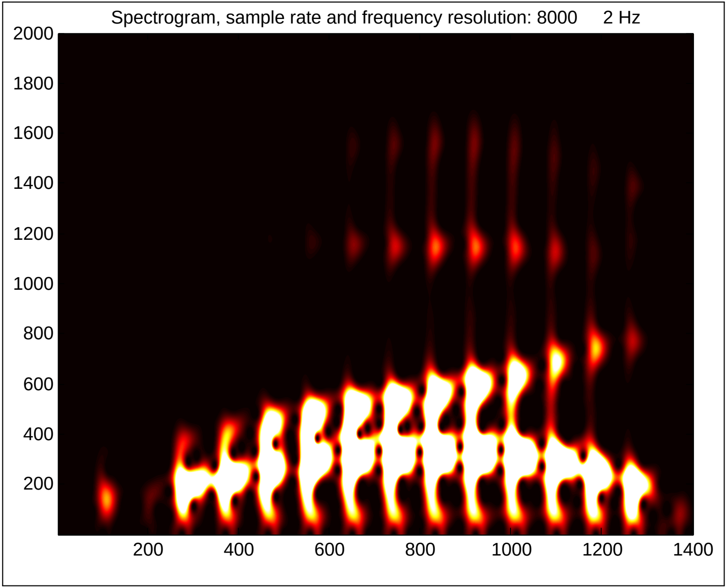
“Why?”: Positive part of Born–Jordan distribution



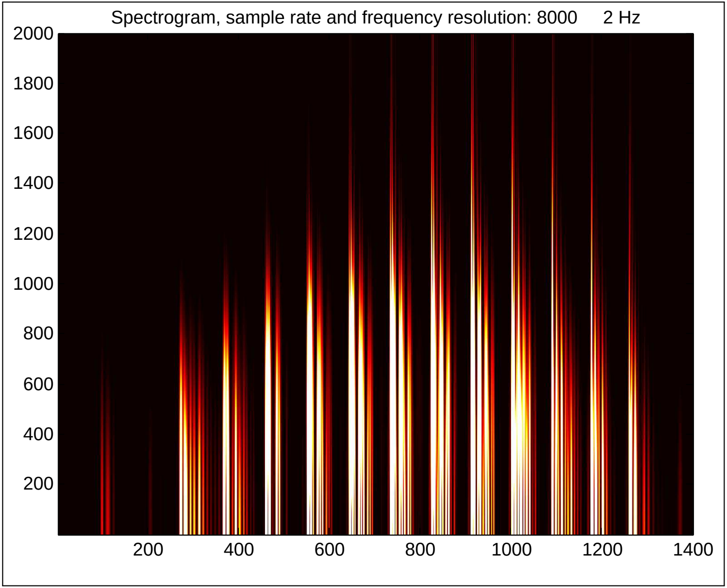
"Why?": spectrogram 1/3



“Why?”: spectrogram 2/3



“Why?”: spectrogram 3/3



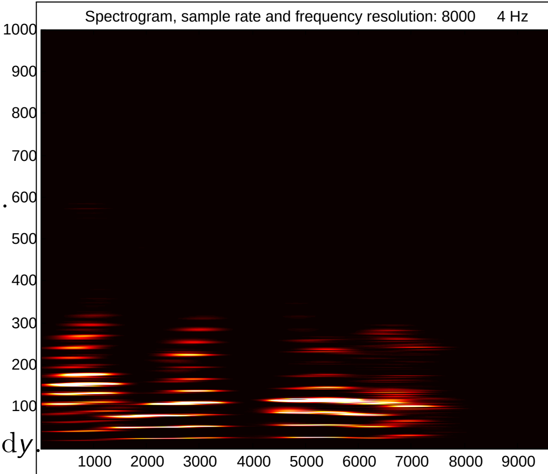
Fourier transform, spectrogram

Test function $u \in \mathcal{S}(\mathbb{R})$,
Fourier transform $\widehat{u} \in \mathcal{S}(\mathbb{R})$,

$$\widehat{u}(\eta) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\, u(y)\, \mathrm{d}y.$$

Spectrogram $|G(u,w)|^2$, where
w-windowed Fourier transform

$$G(u,w)(x,\eta) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\, u(y)\, w(y-x)^* \mathrm{d}y.$$



“Big picture”: sharp time-frequency localization!

It is beneficial to understand the connection

$$\begin{aligned} & \textit{Time}-\textit{frequency distributions} \\ \leftrightarrow\quad & \textit{pseudodifferential quantizations}, \end{aligned}$$

both in Fourier analysis
and in real-life applications.

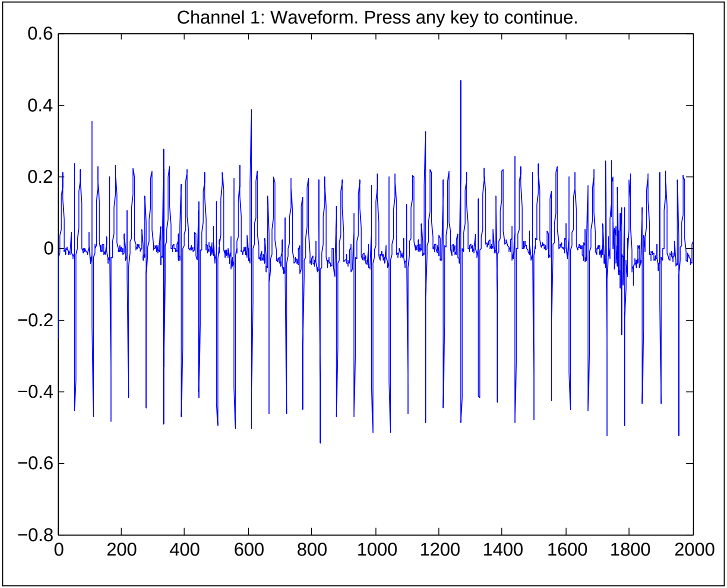
For signals $u : \mathbb{R} \to \mathbb{C}$ there exists a unique
dilation-invariant time-local time-frequency distribution

$$Q[u] = \psi * W(u,u) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$$

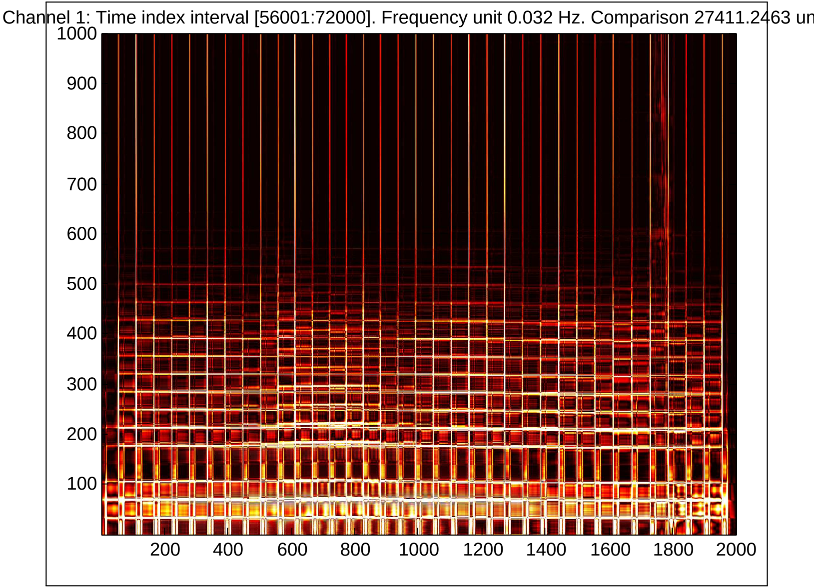
that maps **comb-to-grid** [VT 2014].

This unique entity is called the
Born–Jordan distribution!

EKG (Electro-KardioGram)



EKG: Born–Jordan distribution (absolute value)



Time-frequency world $\leftrightarrow$ Quantization

Wigner transform $W(u,v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ of signals $u, v : \mathbb{R} \to \mathbb{C}$:

$$W(u,v)(x,\eta) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\ u(x+y/2)\ v(x-y/2)^*\ \mathrm{d}y.$$

A chosen Cohen class time-frequency transform

$$\psi * W(u,v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$$

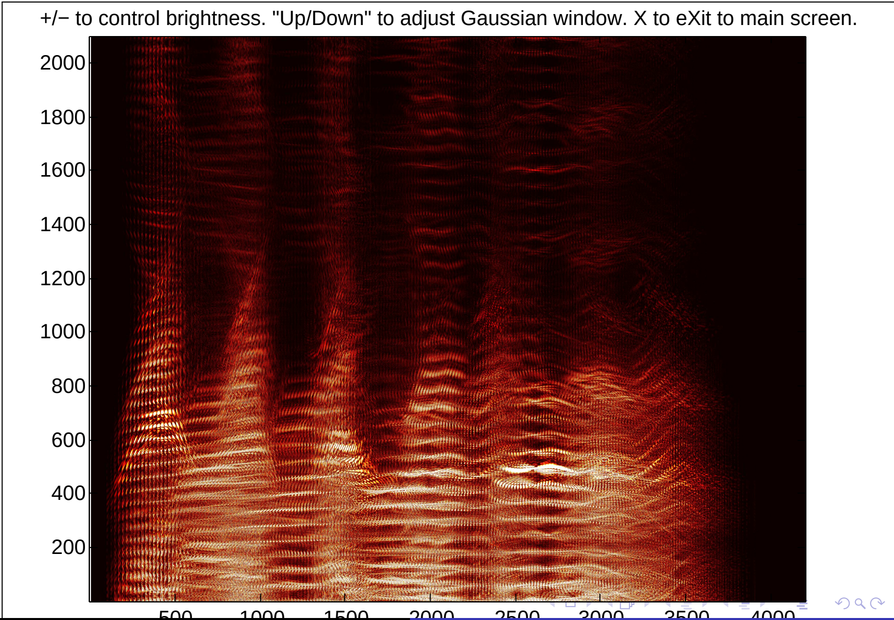
defines the ψ-quantization $\sigma \mapsto A = A_{\psi,\sigma}$ by

$$\langle u, Av\rangle_{L^2(\mathbb{R})} := \langle \psi * W(u,v), \sigma\rangle_{L^2(\mathbb{R}\times\widehat{\mathbb{R}})}.$$

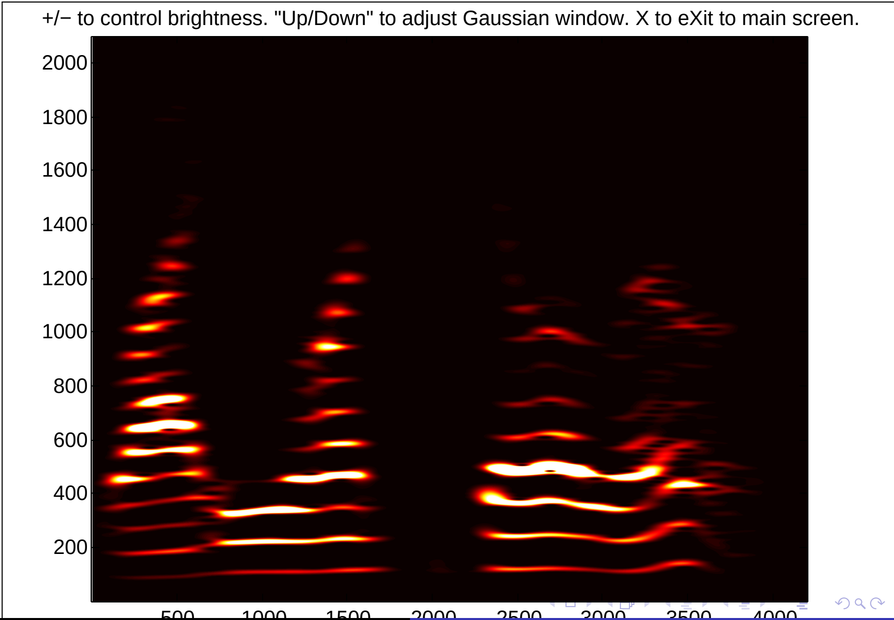
Here $\sigma : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ is called the ψ-symbol of the operator A.

Example. $\widehat{\psi}(\xi, y) = \mathrm{sinc}(\xi \cdot y)$ for Born–Jordan.
Example. Spectrogram $|G(u,w)|^2 = \psi * W(u,u)$, where $\psi = W(w,w)$.

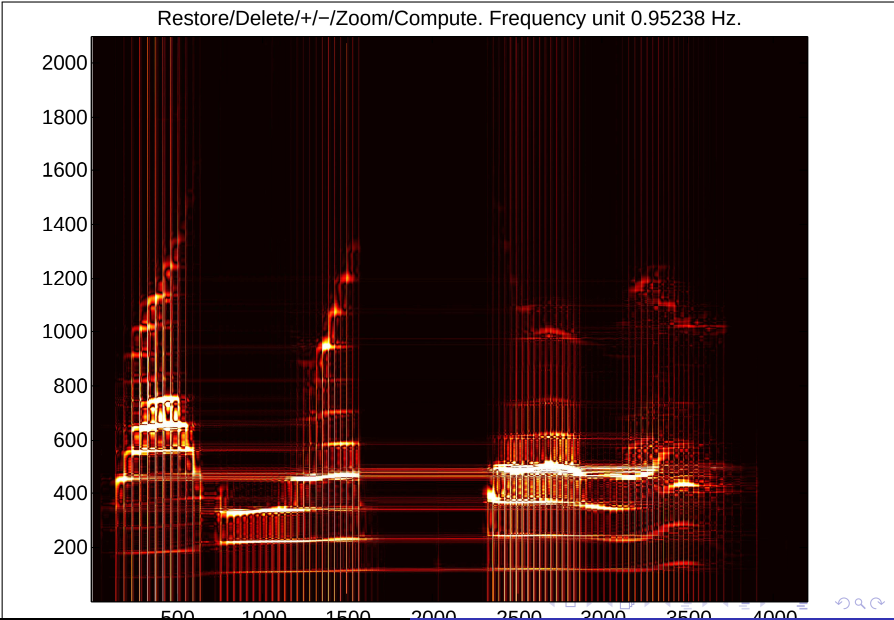
Wigner distribution (absolute value)



Spectrogram (with a Gaussian window)



Born-Jordan distribution (absolute value)



Born-Jordan quantization $\sigma \mapsto A_\sigma$

Born–Jordan transform $Q(u,v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$,

$$Q(u,v)(x,\eta) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta} \left[\frac{1}{y} \int_{x-y/2}^{x+y/2} u(t+y/2)\ v(t-y/2)^*\ \mathrm{d}t \right] \mathrm{d}y.$$

Born-Jordan quantization $\sigma \mapsto A_\sigma$ defined by

$$\langle u, A_\sigma v\rangle_{L^2(\mathbb{R})} = \langle Q(u,v), \sigma\rangle_{L^2(\mathbb{R}\times\widehat{\mathbb{R}})}$$

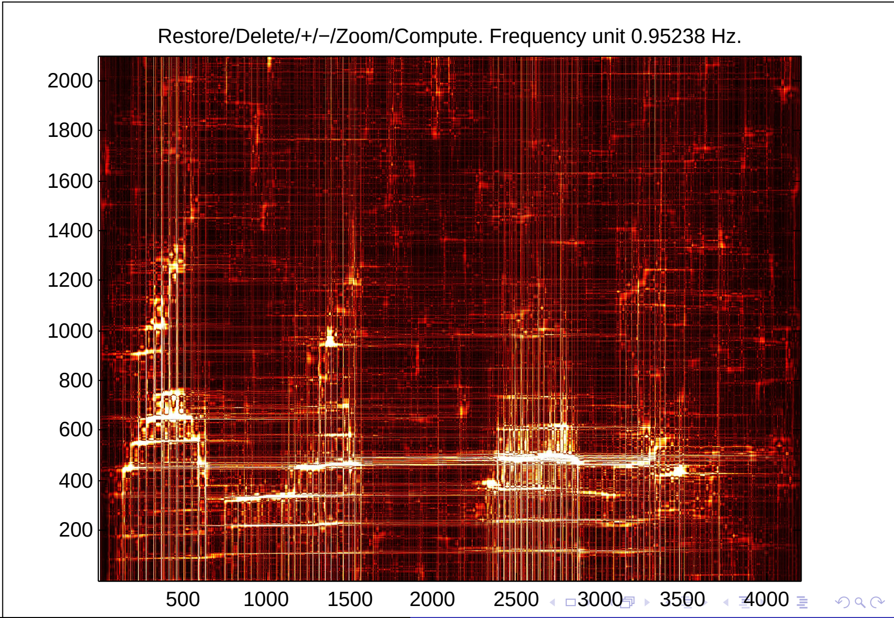
means

$$A_\sigma v(x) = \int_{\mathbb{R}} K(x,y)\ v(y)\ \mathrm{d}y$$

with the Schwartz kernel

$$K(x,y) = \int_{\widehat{\mathbb{R}}} \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta} \left[\frac{1}{y-x} \int_x^y \sigma(t,\eta)\ \mathrm{d}t \right] \mathrm{d}\eta.$$

Time-frequency distribution (noisy signal v)...



Time-frequency distribution (filtered signal $A_\sigma v$)