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REGULARITY PROPERTIES OF SOLUTIONS OF FRACTIONAL EVOLUTION EQUATIONS

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Abstract: The equation

$$D_t^\alpha(u_t - u_1) + Bu(t) = f(t), \quad u(0) = u_0, \quad t \geq 0, \tag{$*$}$$

is considered in a Banach space X. In $(*)$, D_t^α denotes the fractional derivative of order $\alpha \in (0,1)$; thus $(*)$ is of order $1+\alpha$, with $u_t(0) = u_1$. It is assumed that B is a closed, positive, linear map of $\mathcal{D}(B) \subset X$ into X, that $f \in \mathcal{C}([0,T];X)$ and that $u_0, u_1 \in X$. Conditions implying that Bu is Hölder-continuous, or takes values in some interpolation space determined by B, are given.

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1. Introduction and main results.

In this paper we continue our work on regularity properties of solutions of fractional evolution equations. In particular, we study the equation

$$D_t^\alpha(u_t - u_1)(t) + Bu(t) = f(t), \quad u(0) = u_0, \quad t \geq 0, \tag{1}$$

which is of order $1 + \alpha$. The function u is the unknown, taking values in a Banach space X; $\alpha \in (0,1)$; u_0, u_1 and f are given, with $u_0, u_1 \in X$ and $f \in \mathcal{C}([0,T]; X)$ for some $T > 0$.

In (1), D_t^α denotes the fractional derivative of order α, i.e.,

$$(D_t^\alpha v)(t) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g_{1-\alpha}(t-s) v(s)\,\mathrm{d}s, \quad t > 0,$$
$$(D_t^\alpha v)(0) \stackrel{\text{def}}{=} \lim_{h \downarrow 0} \frac{1}{h} \int_0^h g_{1-\alpha}(h-s) v(s)\,\mathrm{d}s,$$

where

$$g_\beta(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\beta)} t^{\beta-1}, \quad t > 0, \quad \beta > 0,$$

and where v is (at least) continuous and satisfies $v(0) = 0$.

The operator B is taken to be a closed (not necessarily densely defined) linear map of $\mathcal{D}(B) \subset X$ into X. Thus $\mathcal{D}(B)$ is a Banach space equipped with the usual graph norm. We assume B to be positive, i.e., that the resolvent set of $-B$ contains $\mathbb{R}^+ = [0, \infty)$, and that

$$\sup_{\lambda \geq 0} \|(\lambda + 1)(\lambda I + B)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

For $\omega \in [0, \pi)$, we define

$$\Sigma_\omega \stackrel{\text{def}}{=} \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \omega \}.$$

We recall that, if B is positive, then there exists a number $\eta \in (0, \pi)$ such that $\rho(-B) \supset \overline{\Sigma_\eta}$ and

$$\sup_{\lambda \in \overline{\Sigma_\eta}} \|(\lambda + 1)(\lambda I + B)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{2}$$

The spectral angle of B is defined by

$$\phi_B \stackrel{\text{def}}{=} \inf\{ \omega \in (0,\pi) \mid \rho(-B) \supset \overline{\Sigma_{\pi-\omega}} \text{ and } \sup_{\lambda \in \overline{\Sigma_{\pi-\omega}}} \|(\lambda+1)(\lambda I + B)^{-1}\|_{\mathcal{L}(X)} < \infty \}.$$

In a previous paper, [1], we examined the equation

$$D_t^\beta(u - u_0) + Bu = f, \qquad \beta \in (0,1), \tag{3}$$

under analogous assumptions on B, u_0 and f, and obtained maximal regularity results in certain interpolation spaces. As to the assumptions on u_0, they were shown to be both necessary and sufficient.

In this paper, we extend these studies to the case $\beta \in (1,2)$. We mainly consider sufficient conditions for the solutions of (1) to be smooth and will return to necessary hypotheses in later work. Only at the end do we here give some brief comments on the converse analysis of (1). However, in the case where either u_0 or u_1 vanishes, our results are optimal. We write $D_t^\beta = D_t^\alpha D_t$, with $\beta = \alpha + 1$, $\alpha \in (0,1)$, and include the initial value of u_t in the convolution integral.

The case $\alpha = 0$, i.e., the differential equation

$$u_t + Bu = f, \quad u(0) = u_0, \quad t \geq 0,$$

was considered by Sinestrari [13], and by Da Prato and Sinestrari [5]. Their results provided partial motivation for [1]. Hölder-regularity results for (1) with $\alpha \in (0,1)$ have previously been obtained by Da Prato, Iannelli and Sinestrari [4]. We comment briefly on their results below, at the end of this Section.

In forthcoming work, we will apply the results given here to fractional partial differential equations and extend them to cases with nonconstant Hölder continuous coefficients. For (3), and with B a spatial derivative of order ≤ 1, this was done in [1] and [2].

Our analysis concerns strict solutions of (1). With $f \in \mathcal{C}([0,T];X)$; $u_0, u_1 \in X$, and $\alpha \in (0,1)$, these are defined as follows:

Definition 1. *A function $u : [0,T] \to X$ is said to be a strict solution of* (1) *on* $[0,T]$ *if* $u, u_t \in \mathcal{C}([0,T];X)$, $u \in \mathcal{C}([0,T];\mathcal{D}(B))$, $u(0) = u_0$, $g_{1-\alpha} * (u_t - u_1) \in \mathcal{C}^1([0,T];X)$, *($*$ denotes convolution) and* (1) *holds for all* $t \in [0,T]$.

We summarize our results in Theorem 2 below. In this Theorem, we formulate some existence, uniqueness and regularity results on strict solutions of (1).

Concerning the interpolation spaces determined by an operator B, we use the notation (here $\gamma \in (0,1]$ and $p \in [1,\infty]$)

$$\mathcal{D}_B(\gamma,p) \stackrel{\text{def}}{=} (X, \mathcal{D}(B))_{\gamma,p},$$
$$\mathcal{D}_B(\gamma) \stackrel{\text{def}}{=} (X, \mathcal{D}(B))_{\gamma}.$$

By [6, Thm. 3.1, p. 159] and [7, p. 314] one has the following characterization of $\mathcal{D}_B(\gamma,\infty)$ and $\mathcal{D}_B(\gamma)$: If η is some number such that $0 \leq \eta < \pi - \phi_B$, then

$$\mathcal{D}_B(\gamma,\infty) = \{\, x \in X \mid \sup_{\substack{|\arg\lambda| \leq \eta \\ \lambda \neq 0}} \|\lambda^\gamma B(\lambda I + B)^{-1} x\|_X < \infty \,\},$$
$$\mathcal{D}_B(\gamma) = \{\, x \in \mathcal{D}_B(\gamma,\infty) \mid \lim_{\substack{|\lambda| \to \infty \\ |\arg\lambda| \leq \eta}} \|\lambda^\gamma B(\lambda I + B)^{-1} x\|_X = 0 \,\}.$$

The Hölder spaces $\mathcal{C}^\gamma$, $0 < \gamma < 1$, are defined by

$$\mathcal{C}^\gamma([0,T];X) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{C}([0,T];X) \;\middle|\; \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma} < \infty \right\}$$

with

$$\|f\|_{\mathcal{C}^\gamma} \stackrel{\text{def}}{=} \sup_{t \in [0,T]} \|f(t)\|_X + \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma}.$$

If $\gamma \in (1,2)$, then $\mathcal{C}^\gamma \stackrel{\text{def}}{=} \{ f \mid f' \in \mathcal{C}^{\gamma-1} \}$. The little Hölder spaces h^γ, $\gamma \in (0,1)$, are defined by

$$h^\gamma([0,T];X) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{C}^\gamma([0,T];X) \;\middle|\; \limsup_{\substack{\delta \downarrow 0 \\ t,s\in[0,T],0<|t-s|\le\delta}} \frac{\|f(t)-f(s)\|_X}{|t-s|^\gamma} = 0 \right\}.$$

If $\gamma \in (1,2)$, then $h^\gamma \stackrel{\text{def}}{=} \{ f \mid f' \in h^{\gamma-1} \}$.

Theorem 2. *Suppose*

(i) $\alpha \in (0,1)$;

(ii) B *is a positive operator in a complex Banach space* X *with spectral angle* $\phi_B < \pi(\frac{1}{2} - \frac{\alpha}{2})$;

(iii) $u_0 \in \mathcal{D}(B)$, $u_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$;

(iv) $f \in \mathcal{C}([0,T];X)$, *where* $T > 0$.

Then the following statements hold:

(a) *Let* $\gamma \in (0,1)$ *and* $f \in \mathcal{C}^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *if both*

$$Bu_0 - f(0) \in \mathcal{D}_B\Big(\frac{\gamma}{1+\alpha}, \infty\Big), \tag{4}$$

and

$$u_1 \in \mathcal{D}_B\Big(\frac{\alpha+\gamma}{1+\alpha}, \infty\Big) \tag{5}$$

hold. Moreover, in this case there is a constant $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bu(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)}$$
$$\le M\Bigg(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\frac{\gamma}{1+\alpha},\infty)} + \|u_1\|_{\mathcal{D}_B(\frac{\gamma+\alpha}{1+\alpha},\infty)} + \|f(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)} \Bigg).$$

(b) *Let* $\gamma \in (1, 1+\alpha]$ *and* $f \in \mathcal{C}^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *if both* (4) *and*

$$u_1 - B^{-1}f'(0) \in \mathcal{D}_{B^2}\Big(\frac{\alpha+\gamma}{2(1+\alpha)}, \infty\Big) \tag{6}$$

hold. Moreover, in this case there is a constant $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bu(\underline{t}) - f(0) - \underline{t}f'(0)\|_{\mathcal{C}^\gamma([0,T];X)} \le M\Bigg(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\frac{\gamma}{1+\alpha},\infty)} +$$
$$\|u_1 - B^{-1}f'(0)\|_{\mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)},\infty)} + \|f(\underline{t}) - f(0) - \underline{t}f'(0)\|_{\mathcal{C}^\gamma([0,T];X)} \Bigg).$$

(c) *Let* $\gamma \in (1+\alpha, 2)$ *and* $f \in \mathcal{C}^\gamma([0,T];X)$. *Then there exists a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *if both* $Bu_0 = f(0)$ *and*

(6) *hold. Moreover, in this case there is a constant* $M = M(\gamma, \alpha, B, T)$ *such that*

$$\|Bu(\underline{t}) - f(0) - \underline{t}f'(0)\|_{\mathcal{C}^\gamma([0,T];X)}$$
$$\leq M\left(\|u_1 - B^{-1}f'(0)\|_{\mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)},\infty)} + \|f(\underline{t}) - f(0) - \underline{t}f'(0)\|_{\mathcal{C}^\gamma([0,T];X)}\right).$$

(d) *Let* $\gamma \in (0,1]$ *and* $f \in \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in Z \stackrel{\text{def}}{=} \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *if both*

$$Bu_0 - f(0) \in \mathcal{D}_B(\gamma,\infty)$$

and

$$u_1 \in \mathcal{D}(B^{\frac{\alpha}{1+\alpha}}), \qquad B^{\frac{\alpha}{1+\alpha}}u_1 \in \mathcal{D}_B(\gamma,\infty), \tag{7}$$

hold. Moreover, in this case there is a constant $M = M(\gamma, \alpha, B, T)$ *such that*

$$\|Bu(\underline{t}) - f(0)\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \leq M\left(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\gamma,\infty)}\right.$$
$$\left. + \|B^{\frac{\alpha}{1+\alpha}}u_1\|_{\mathcal{D}_B(\gamma,\infty)} + \|f(\underline{t}) - f(0)\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))}\right).$$

(e) *Let* $\gamma \in (0,1)$ *and* $f \in h^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in h^\gamma([0,T];X)$ *if both* $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{1+\alpha})$ *and* $u_1 \in \mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha})$ *hold.*

(f) *Let* $\gamma \in (1, 1+\alpha)$ *and* $f \in h^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in h^\gamma([0,T];X)$ *if both* $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{1+\alpha})$ *and*

$$u_1 - B^{-1}f'(0) \in \mathcal{D}_{B^2}\left(\frac{\alpha+\gamma}{2(1+\alpha)}\right) \tag{8}$$

hold.

(g) *Let* $\gamma \in [1+\alpha, 2)$ *and* $f \in h^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in h^\gamma([0,T];X)$ *if both* $Bu_0 = f(0)$ *and* (8) *hold.*

(h) *Let* $\gamma \in (0,1)$ *and* $f \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$. *Then there is a unique strict solution* u *of* (1) *satisfying* $Bu(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *if both* $Bu_0 - f(0) \in \mathcal{D}_B(\gamma)$ *and*

$$u_1 \in \mathcal{D}(B^{\frac{\alpha}{1+\alpha}}), \quad B^{\frac{\alpha}{1+\alpha}}u_1 \in \mathcal{D}_B(\gamma) \tag{9}$$

hold.

The conditions (7) and (9) may be written

$$u_1 \in \mathcal{D}_{B^2}\left(\frac{\alpha+\gamma+\alpha\gamma}{2(1+\alpha)},\infty\right), \qquad u_1 \in \mathcal{D}_{B^2}\left(\frac{\alpha+\gamma+\alpha\gamma}{2(1+\alpha)}\right),$$

respectively.

For the definitions and properties of fractional powers of densely defined operators we refer to [1, Theorem 10], [15, p. 98–103]. For powers of non-densely defined operators, see [9, p. 54]. Observe that by (ii) of Theorem 2, the operator B is sectorial in the sense of [9]. Also note that the statements of [1, Theorem 10] can be applied to the operator B.

To solve (1) we write, in the cases where $\gamma \in (0,1]$, i.e., in cases (a), (d), (e), (h),

$$u(t) = v(t) + w(t) + z(t) + B^{-1}f(0), \tag{10}$$

where v, w, z solve, respectively,

$$D_t^\alpha v_t + Bv = 0, \quad v(0) = u_0 - B^{-1}f(0), \quad v_t(0) = 0, \tag{11}$$

$$D_t^\alpha (w_t - u_1) + Bw = 0, \quad w(0) = 0, \quad w_t(0) = u_1, \tag{12}$$

$$D_t^\alpha z_t + Bz = f(t) - f(0), \quad z(0) = z_t(0) = 0. \tag{13}$$

Then u, defined by (10), solves (1).

If $\gamma \in (1,2)$, i.e., in cases (b), (c), (f), (g), we write

$$u(t) = v(t) + w(t) + z(t) + B^{-1}f(0) + tB^{-1}f'(0), \tag{14}$$

where v, w, z solve, respectively, (11) and

$$D_t^\alpha \Big(w_t - [u_1 - B^{-1}f'(0)]\Big) + Bw = 0, \quad w(0) = 0, \quad w_t(0) = u_1 - B^{-1}f'(0), \tag{15}$$

$$D_t^\alpha z_t + Bz = f(t) - f(0) - tf'(0), \quad z(0) = z_t(0) = 0. \tag{16}$$

Then u, defined by (14), solves (1).

First, in Section 2, existence and regularity for solutions of (11) is given. These results are obtained by the resolvent approach. Lemma 3 gives the required properties of the resolvent associated with (11). Lemma 4 gives necessary and sufficient conditions on $v(0)$ for $Bv(t)$ to be either Hölder-continuous, or bounded in an interpolation space determined by B.

Next, the resolvent approach is used to analyze solutions of (12) (and (15)). Lemma 5 gives the properties of the associated resolvent; in Lemma 6 we formulate necessary and sufficient conditions on $w_t(0)$ for Bw to be of a specified type.

The proofs of Lemma 3 and of Lemma 5 are given in Section 4.

To analyze (13) and (16), we use the method of sums (see Lemma 7) of Da Prato and Grisvard [3]. For the application of this method, we need some results (formulated in Lemma 8) on fractional derivatives of order between 1 and 2. For derivatives of order between zero and one, the corresponding results were given in [1]. The regularity results on (13) and (16) are given in Lemma 9.

Theorem 2 is an immediate consequence of Lemmas 4, 6, and 9.

In the last Section, we comment briefly on the converse analysis. In particular, a required interpolation statement is formulated.

Regularity results for

$$u_t + B(k * u)(t) = f(t), \quad u(0) = u_0, \tag{17}$$

with (typically) $k(t) = ct^{-\beta}$, $\beta \in (0,1)$, have been obtained in [4]. To compare results, invert (17) to obtain,

$$D_t^\alpha u_t + Bu = h \overset{\text{def}}{=} \frac{d}{dt}(t^{-\alpha} * f),$$

where $\alpha = 1 - \beta$. To have h Hölder-continuous, take $f(t) = ct^\rho$ for some $\rho \in (\alpha, 1)$. Then $h \in \mathcal{C}^\gamma([0,T];X)$, with $\gamma = \rho - \alpha$; $f(0) = h(0) = u_t(0) = 0$. By [4, Theorem 4.4], we have that $Bu_0 \in \mathcal{D}_B(\frac{\gamma+\alpha}{1+\alpha}, \infty)$ implies $D_t^\alpha u_t \in \mathcal{C}^\gamma([0,T];X)$, hence $Bu \in \mathcal{C}^\gamma([0,T];X)$. Our Lemma 4 (a) does, however, give the stronger result that in this case $Bu_0 \in \mathcal{D}_B(\frac{\gamma}{1+\alpha}, \infty)$ is equivalent to $Bu \in \mathcal{C}^\gamma([0,T];X)$.

2. **Results on homogeneous equations.**

We begin by considering (11) and employ the resolvent associated to this equation. Thus, we set

$$\begin{aligned} S(t)v_0 &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1,\theta}} \mathrm{e}^{\lambda t} \lambda^\alpha (\lambda^{1+\alpha} I + B)^{-1} v_0 \, \mathrm{d}\lambda, \quad t > 0, \\ S(0)v_0 &= v_0, \end{aligned} \tag{18}$$

where

$$\theta \in \left(\frac{\pi}{2}, \frac{\pi - \phi_B}{1+\alpha} \right), \tag{19}$$

and

$$\Gamma_{r,\theta} \overset{\text{def}}{=} \{ r\mathrm{e}^{\mathrm{i}t} \mid |t| \le \theta \} \cup \{ \rho \mathrm{e}^{\mathrm{i}\theta} \mid r < \rho < \infty \} \cup \{ \rho \mathrm{e}^{-\mathrm{i}\theta} \mid r < \rho < \infty \}. \tag{20}$$

Formally, $\underline{\lambda}^\alpha(\underline{\lambda}^{1+\alpha} I + B)^{-1} v_0$ is the Laplace transform of the solution v of (11). We have the following result on $S(t)$.

Lemma 3. *Assume that* (i) *and* (ii) *of Theorem 2, and* (19) *hold. Define S by* (18), *let k be a nonnegative integer, and let $l = 0, 1$. Then*

(a) *$S(t) \in \mathcal{L}(X)$ for each $t \ge 0$,*
(b) *$S(\bullet) \in \mathcal{C}^\infty((0,\infty); \mathcal{L}(X))$.*
(c) *For $t > 0$, the range of $S(t)$ is contained in $\mathcal{D}(B)$.*
(d) *For $t > 0$, $BS^{(k)}(t) \in \mathcal{L}(X)$; $BS^{(k)}(\bullet) \in \mathcal{C}^\infty((0,\infty); \mathcal{L}(X))$.*
(e) *For any fixed $\theta \in (0, \frac{\pi - \phi_B}{1+\alpha} - \frac{\pi}{2})$, there exists an analytic extension of $B^l S^{(k)}(\bullet)$ to the sector $|\arg z| \le \theta$.*
(f) $\sup_{t>0} \| t^{l(1+\alpha)+k} B^l S^{(k)}(t) \|_{\mathcal{L}(X)} < \infty$.
(g) *For $v_0 \in X$, $v(t) \overset{\text{def}}{=} S(t)v_0$, $t \ge 0$, one has that v is a strict solution of*

$$D_t^\alpha v_t + Bv = 0, \quad v(0) = v_0, \quad v'(0) = 0, \quad t \in [0,T], \tag{21}$$

for any $T > 0$, iff $v_0 \in \mathcal{D}(B)$ and $Bv_0 \in \overline{\mathcal{D}(B)}$.
(h) *If v is a strict solution of* (21), *then $v, v_t, D_t^\alpha v_t, Bv \in \mathcal{BC}(\mathbb{R}^+; X)$.*

For the proof of Lemma 3, see Section 4.

For the solutions of (11) we have the following regularity results.

Lemma 4. *Let* (i) *and* (ii) *of Theorem 2 hold. Let* $v_0 \in \mathcal{D}(B)$ *and* $v(\underline{t}) \stackrel{\text{def}}{=} S(\underline{t})v_0$. *Then the following conclusions hold for each* $T > 0$:

(a) *Let* $\gamma \in (0, 1+\alpha]$, $\gamma \neq 1$. *Then* $Bv(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{1+\alpha}, \infty)$. *Moreover, in this case there is a constant* $M = M(\gamma, \alpha, B)$, *but independent of* T, *such that*

$$\|Bv(\underline{t})\|_{\mathcal{C}^\gamma([0,T];X)} \leq M\|Bv_0\|_{\mathcal{D}_B(\frac{\gamma}{1+\alpha},\infty)}. \tag{22}$$

(b) *Let* $\gamma \in (1+\alpha, 2)$. *Then* $Bv(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $v(\underline{t}) = v(0)$ *(iff* $Bv_0 = 0$*).*

(c) *Let* $\gamma \in (0,1]$. *Then* $Bv(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B)$, *but independent of* T, *such that*

$$\|Bv(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \leq M\|Bv_0\|_{\mathcal{D}_B(\gamma,\infty)}. \tag{23}$$

(d) *Let* $\gamma \in (0, 1+\alpha)$, $\gamma \neq 1$. *Then* $Bv(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{1+\alpha})$.

(e) *Let* $\gamma \in [1+\alpha, 2)$. *Then* $Bv(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $v(\underline{t}) = v_0$ *(iff* $Bv_0 = 0$*).*

(f) *Let* $\gamma \in (0,1)$. *Then* $Bv(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $Bv_0 \in \mathcal{D}_B(\gamma)$.

Observe that in all cases of Lemma 4, the hypotheses made and Lemma 3 give that $v(t)$ is a strict solution of $D_t^\alpha v_t + Bv = 0$, $v(0) = v_0$, $v_t(0) = 0$.

Except for some technicalities, the proof of Lemma 4 parallels that of [1, Lemma 12]. Therefore, we omit the proof.

For the analysis of (12) and (15) we need the corresponding resolvent. We set, for $y \in X$,

$$\begin{aligned} S_1(t)y &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1,\theta}} \mathrm{e}^{\lambda t} \lambda^{-1+\alpha} (\lambda^{1+\alpha} I + B)^{-1} y \, \mathrm{d}\lambda, \quad t > 0, \\ S_1(0)y &= 0, \end{aligned} \tag{24}$$

where the integration path is as in (18). One then has the following result.

Lemma 5. *Assume that* (i) *and* (ii) *of Theorem 2, and* (19) *hold. Define* S_1 *by* (24), *and let* k *be a nonnegative integer. Then properties* (a)–(e) *of Lemma 3 hold with* S *replaced by* S_1. *Moreover, for* $l = 0, 1$,

$$\sup_{t>0} \|t^{l(1+\alpha)+k-1} B^l S_1^{(k)}(t)\|_{\mathcal{L}(X)} < \infty. \tag{25}$$

The function $w(t) \stackrel{\text{def}}{=} S_1(t)w_1$ *is a strict solution of*

$$D_t^\alpha(w_t - u_1) + Bw = 0, \quad w(0) = 0, \quad w_t(0) = w_1, \quad t \geq 0, \tag{26}$$

iff $w_1 \in \mathcal{D}_B\left(\frac{\alpha}{1+\alpha}\right)$.

If w *is strict solution of* (26), *then* w, w_t, $Bw \in \mathcal{BC}(\mathbb{R}^+;X)$.

For the proof of Lemma 5, see Section 4.

Concerning the solutions $w(t)$ of (26), we then have the following regularity statements.

Lemma 6. *Let* (i) *and* (ii) *of Theorem 2 hold, let* $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$ *and* $w(t) \stackrel{\text{def}}{=} S_1(t)w_1$. *Then the following statements hold for each* $T > 0$:

(a) *Let* $\gamma \in (0,2)$, $\gamma \neq 1$. *Then* $Bw(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $w_1 \in \mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)},\infty)$. *Moreover, in this case there exists a constant* $M = M(\gamma,\alpha,B)$, *independent of* T, *such that*

$$||Bw(\underline{t})||_{\mathcal{C}^\gamma([0,T];X)} \leq M||w_1||_{\mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)},\infty)}. \tag{27}$$

(b) *Let* $\gamma \in (0,1)$. *Then* $Bw(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $w_1 \in \mathcal{D}(B^{\frac{\alpha}{1+\alpha}})$ *and* $B^{\frac{\alpha}{1+\alpha}}w_1 \in \mathcal{D}_B(\gamma,\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B)$, *independent of* T, *such that*

$$||Bw(\underline{t})||_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma;\infty))} \leq M||B^{\frac{\alpha}{1+\alpha}}w_1||_{\mathcal{D}_B(\gamma,\infty)}. \tag{28}$$

(c) *Let* $\gamma \in (0,2)$, $\gamma \neq 1$. *Then* $Bw(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $w_1 \in \mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)})$.

(d) *Let* $\gamma \in (0,1)$. *Then* $Bw(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $w_1 \in \mathcal{D}(B^{\frac{\alpha}{1+\alpha}})$ *and* $B^{\frac{\alpha}{1+\alpha}}w_1 \in \mathcal{D}_B(\gamma)$.

In all cases of Lemma 6, the assumptions made and Lemma 5 imply that $w(t)$ is a strict solution of (26). For the proof of Lemma 6, see Section 5.

3. Method of sums and nonhomogeneous results.

To analyze

$$D_t^\alpha z_t + Bz = h, \quad z_t(0) = z(0) = 0, \tag{29}$$

with $h \neq 0$, we use the method of sums of Da Prato and Grisvard [3]. The following Lemma [1, Theorem 8] reformulates [3, Theorem 3.11].

Lemma 7. *Assume that* $\tilde{X}$ *is a complex Banach space and that* $\tilde{A}$, $\tilde{B}$ *are resolvent commuting positive operators in* $\tilde{X}$ *with spectral angles* $\phi_{\tilde{A}}$ *and* $\phi_{\tilde{B}}$, *respectively, such that* $\phi_{\tilde{A}} + \phi_{\tilde{B}} < \pi$. *If* $\tilde{Y}$ *is one of the spaces* $\mathcal{D}_{\tilde{A}}(\gamma,p)$, $\mathcal{D}_{\tilde{A}}(\gamma)$, $\mathcal{D}_{\tilde{B}}(\gamma,p)$ *or* $\mathcal{D}_{\tilde{B}}(\gamma)$, *where* $\gamma \in (0,1]$ *and* $p \in [1,\infty]$, *and if* $y \in \tilde{Y}$, *then there is a unique* $x \in \mathcal{D}(\tilde{A}) \cap \mathcal{D}(\tilde{B})$ *such that* $\tilde{A}x + \tilde{B}x = y$. *Moreover,* $\tilde{A}x$ *and* $\tilde{B}x \in Y$ *and there exists a constant* C *such that* $||\tilde{A}x||_{\tilde{Y}} + ||\tilde{B}x||_{\tilde{Y}} \leq C||y||_{\tilde{Y}}$.

To apply this Lemma, we write $\tilde{X} \stackrel{\text{def}}{=} \mathcal{C}_0([0,T];X) \stackrel{\text{def}}{=} \{\, u \in \mathcal{C}([0,T];X) \,|\, u(0) = 0\,\}$ and define $\tilde{B}$ in $\tilde{X}$ by

$$\mathcal{D}(\tilde{B}) = \mathcal{C}_0([0,T];\mathcal{D}(B)), \quad (\tilde{B}u)(\underline{t}) = Bu(\underline{t}), \quad u \in \mathcal{D}(\tilde{B}).$$

Then $\tilde{B}$ is a positive operator in $\tilde{X}$ with spectral angle $\phi_{\tilde{B}} = \phi_B$. For the interpolation spaces of $\tilde{B}$ one has, $\gamma \in (0,1]$,

$$\mathcal{D}_{\tilde{B}}(\gamma;\infty) = \tilde{X} \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty)), \quad \mathcal{D}_{\tilde{B}}(\gamma) = \mathcal{C}_0([0,T];\mathcal{D}_B(\gamma)).$$

We consider the fractional derivative as an operator $\tilde{A}_{1+\alpha}$ in $\tilde{X}$ by

$$\mathcal{D}(\tilde{A}_{1+\alpha}) = \{\, u \in \tilde{X} \mid u_t \in \tilde{X}, g_{1-\alpha} * u_t \in \mathcal{C}^1([0,T];X), (D_t^\alpha u_t)(0) = 0 \,\},$$
$$(\tilde{A}_{1+\alpha}u)(\underline{t}) \stackrel{\text{def}}{=} D_t^\alpha u_t(\underline{t}), \quad u \in \mathcal{D}(\tilde{A}_{1+\alpha}), \quad \alpha \in (0,1).$$

The equation (29) can then be written $\tilde{A}_{1+\alpha}z + \tilde{B}z = h$. In addition, define $\tilde{A}_1$ by

$$\mathcal{D}(\tilde{A}_1) \stackrel{\text{def}}{=} \{\, u \in \tilde{X} \mid u' \in \tilde{X} \,\}; \quad \tilde{A}_1 u = u', \quad u \in \mathcal{D}(\tilde{A}_1),$$

and $\tilde{A}_\alpha$ for $\alpha \in (0,1)$ by

$$\mathcal{D}(\tilde{A}_\alpha) \stackrel{\text{def}}{=} \{\, u \in \tilde{X} \mid g_{1-\alpha} * u \in \mathcal{C}^1([0,T];X), (D_t^\alpha u)(0) = 0 \,\},$$
$$\tilde{A}_\alpha u = D_t^\alpha u; \quad u \in \mathcal{D}(\tilde{A}_\alpha).$$

Concerning the operators $\tilde{A}_{1+\alpha}$, $\alpha \in [0,1)$, one has

Lemma 8.

(a) $\tilde{A}_{1+\alpha}$ *is a positive, densely defined operator with spectral angle* $\phi_{\tilde{A}_{1+\alpha}} = \frac{\pi}{2}(1+\alpha)$ *and* $\tilde{A}_{1+\alpha} = (\tilde{A}_1)^{1+\alpha}$.

(b) *For* $\eta \in (0,1)$ *and* $(1+\alpha)\eta \in (0,1)$ *one has*
$\mathcal{D}_{\tilde{A}_{1+\alpha}}(\eta,\infty) = \{\, f \mid f \in \mathcal{C}^{(1+\alpha)\eta}([0,T];X), f(0) = 0 \,\}$.
For $\eta \in (0,1)$ *and* $(1+\alpha)\eta \in (1,2)$ *one has*
$\mathcal{D}_{\tilde{A}_{1+\alpha}}(\eta,\infty) = \{\, f \mid f \in \mathcal{C}^{(1+\alpha)\eta}([0,T];X); f(0) = f'(0) = 0 \,\}$.

(c) *For* $\eta \in (0,1)$ *and* $(1+\alpha)\eta \in (0,1) \cup (1,2)$ *one has*
$\mathcal{D}_{\tilde{A}_{1+\alpha}}(\eta) = \mathcal{D}_{\tilde{A}_{1+\alpha}}(\eta,\infty) \cap h^{(1+\alpha)\eta}([0,T];X)$.

The corresponding statements for $\tilde{A}_\alpha$, $0 < \alpha < 1$, were given in [1, Lemma 11].

Proof of Lemma 8. (a) By [1, Theorem 10(b) and Lemma 11], $\tilde{A}_{1+\alpha} = \tilde{A}_\alpha \tilde{A}_1 = (\tilde{A}_1)^\alpha \tilde{A}_1 = (\tilde{A}_1)^{1+\alpha}$. To see that $\phi_{\tilde{A}_{1+\alpha}} \le \frac{\pi}{2}(1+\alpha)$ one applies [10, Prop. 4]. Assume that $\phi_{\tilde{A}_{1+\alpha}} < \frac{\pi}{2}(1+\alpha)$. By [14, Proposition 2.3.2], and as $\tilde{A}_{1+\alpha} = (\tilde{A}_1)^{1+\alpha}$, one then has $\phi_{\tilde{A}_1} < \frac{\pi}{2}$. By this contradiction, $\phi_{\tilde{A}_{1+\alpha}} = \frac{\pi}{2}(1+\alpha)$.

(b) and (c) follow by [1, Theorem 10(c) and Lemma 11] and by the Reiteration Theorem.

A combination of Lemma 7 and Lemma 8 immediately implies (b) and (d) of the following Lemma and also (a), (c), if $\gamma \in (0, 1+\alpha)$, $\gamma \neq 1$.

To prove (a), (c), if $\gamma \in [1+\alpha, 2)$, apply $\tilde{A}_\epsilon$ to (29), with $\epsilon \in (\gamma - 1 - \alpha, \gamma - 1)$, use [1, Theorem 10(c)], and thus reduce the problem to a case already known. □

Lemma 9. *Let* (i) *and* (ii) *of Theorem 2 hold. Then the following is true:*

(a) *Let* $h \in \mathcal{C}^\gamma([0,T];X)$ *with* $\gamma \in (0,2)$, $\gamma \neq 1$. *If* $\gamma \in (0,1)$, *assume* $h(0) = 0$. *If* $\gamma \in (1,2)$, *assume* $h(0) = h'(0) = 0$. *Then there exists a unique strict solution* z *of* (29) *such that* $Bz(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$. *Moreover, in this case there exists a constant* $M = M(\gamma, \alpha, B, T)$ *such that*

$$\|Bz(\underline{t})\|_{\mathcal{C}^\gamma([0,T];X)} \le M \|h\|_{\mathcal{C}^\gamma([0,T];X)}.$$

(b) *Let* $h \in \tilde{X} \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$, *with* $\gamma \in (0,1)$. *Then there exists a unique strict solution* z *of* (29) *such that* $Bz(\underline{t}) \in \tilde{X} \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$||Bz(\underline{t})||_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \le M||h||_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))}.$$

(c) *Let* $h \in h^\gamma([0,T];X)$ *with* $\gamma \in (0,2)$, $\gamma \neq 1$. *If* $\gamma \in (0,1)$, *assume* $h(0)=0$. *If* $\gamma \in (1,2)$, *assume* $h(0) = h'(0) = 0$. *Then there exists a unique solution* z *of* (29) *such that* $Bz(\underline{t}) \in h^\gamma([0,T];X)$.

(d) *Let* $h \in \mathcal{C}_0([0,T];\mathcal{D}_B(\gamma))$, *with* $\gamma \in (0,1)$. *Then there exists a unique solution* z *of* (29) *such that* $Bz(\underline{t}) \in \mathcal{C}_0([0,T];\mathcal{D}_B(\gamma))$.

4. **Proofs of Lemmas 3 and 5.**

Proof of Lemma 3. Properties (a)–(d) and (h) are obvious consequences of (18) and of the fact that $\sup_{\lambda\in\Gamma_{1,\theta}}||B(\lambda^{1+\alpha}I+B)^{-1}||_{\mathcal{L}(X)} < \infty$. To obtain (e), use [12, Theorem 0.1, p.5]. By analyticity, and by an application of Cauchys theorem, one has (f).

To prove (g), first let v be a strict solution of (21). Then $v_0 \in \mathcal{D}(B)$ and $\lim_{t\to 0}||Bv(t) - Bv_0)||_X = 0$. Therefore, using also (h),

$$\lim_{\lambda\to\infty}\Big\|\lambda\int_0^\infty \mathrm{e}^{-\lambda t}\{Bv(t) - Bv_0\}\,dt\Big\|_X = 0.$$

By (18), after some computations, for $\lambda > 0$,

$$\lambda\int_0^\infty \mathrm{e}^{-\lambda t}(Bv(t) - Bv_0)\,dt = -B(\lambda^{1+\alpha}I+B)^{-1}Bv_0.$$

Thus,

$$\lim_{\lambda\to\infty}||B(\lambda I+B)^{-1}Bv_0||_X = 0. \tag{30}$$

Employ [8, Theorem 2.1, p. 289, 291] to conclude that $Bv_0 \in \overline{\mathcal{D}(B)}$.

Conversely, let $v_0 \in \mathcal{D}(B)$, $Bv_0 \in \overline{\mathcal{D}(B)}$. By [8, p. 291], (30) follows. Use this fact, the estimate (2) and the identity ($\rho > 0$)

$$B(\rho\mathrm{e}^{\mathrm{i}\beta}I+B)^{-1}Bv_0 = B(\rho\mathrm{e}^{\mathrm{i}\beta}I+B)^{-1}[\rho\mathrm{e}^{\mathrm{i}\eta}(\rho\mathrm{e}^{\mathrm{i}\eta}I+B)^{-1}Bv_0 + B(\rho\mathrm{e}^{\mathrm{i}\eta}I+B)^{-1}Bv_0]$$

to conclude that

$$\lim_{\lambda\to\infty}||B(\lambda I+B)^{-1}Bv_0||_X = 0, \quad \lambda\in\Gamma_{1,\theta}. \tag{31}$$

By a change of variables and by analyticity, we get from (24),

$$Bv(t) - Bv_0 = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1,\theta}} \mathrm{e}^s s^{-1}B\big((\tfrac{s}{t})^{1+\alpha}I+B\big)^{-1}Bv_0\,ds. \tag{32}$$

Combine (31) and (32) to conclude that $\lim_{t\to 0}||Bv(t) - Bv_0||_X = 0$. From this last fact one may easily deduce that $v(t)$ is a strict solution.

Proof of Lemma 5. The properties (a)–(e) and (25) follow as in the proof of Lemma 3.

To prove that w is a strict solution iff $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$, one argues as in the proof of Lemma 3. First, let w be a strict solution of (26). Then it follows that

$$\lim_{\lambda\to\infty} \Big\|\lambda \int_0^\infty e^{-\lambda t} Bw(t)\, dt\Big\|_X = 0.$$

But $B\tilde{w} = \lambda^{\alpha-1} B(\lambda^{1+\alpha} I + B)^{-1} w_1$, and so

$$\lim_{\lambda\to\infty} \|\lambda^\alpha B(\lambda^{1+\alpha} I + B)^{-1} w_1\|_X = 0,$$

which gives $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$.

Conversely, let $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$. We have, by analyticity, for $t > 0$,

$$\begin{aligned} Bw(t) =& \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{t},\theta}} e^{\lambda t} \lambda^{\alpha-1} B(\lambda^{1+\alpha} I + B)^{-1} w_1\, d\lambda \\ =& \frac{1}{2\pi i} \int_{\Gamma_{1,\theta}} e^s s^{-1} \Big(\frac{s}{t}\Big)^\alpha B\Big(\Big(\frac{s}{t}\Big)^{1+\alpha} I + B\Big)^{-1} w_1\, ds\,, \end{aligned}$$

and so $\lim_{t\to 0} \|Bw(t)\|_X = 0$. Again, this fact allows us to deduce that w is a strict solution.

5. Proof of Lemma 6.

(a) It is convenient to separate the proof in two parts; for $\gamma < 1$ and $\gamma > 1$, respectively.

First, let $\gamma \in (0,1)$. Observe that the first part of the claim is then equivalent to

$$Bw(\underline{t}) \in C^\gamma([0,T];X) \qquad \text{iff} \qquad w_1 \in \mathcal{D}_B\Big(\frac{\alpha+\gamma}{1+\alpha}, \infty\Big).$$

Suppose $w_1 \in \mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha}, \infty)$. By (24), and by the analyticity of the integrand, there follows, for $r > 0$,

$$Bw(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} \lambda^{-1+\alpha} B(\lambda^{1+\alpha} I + B)^{-1} w_1\, d\lambda, \tag{33}$$

and so one has, for $0 < s < t$, by the dominated convergence theorem and using the fact that $\sup_{|arg\lambda| \le \theta} \|B(\lambda^{1+\alpha} I + B)^{-1}\|_{\mathcal{L}(X)} < \infty$,

$$\begin{aligned} Bw(t) - Bw(s) &= \frac{1}{2\pi i} \int_{\Gamma_{0,\theta}} [e^{\lambda t} - e^{\lambda s}] \lambda^{-1-\gamma} \lambda^{\alpha+\gamma} B(\lambda^{\alpha+1} I + B)^{-1} w_1\, d\lambda \\ &= (t-s)^\gamma \frac{1}{2\pi i} \int_{\Gamma_{0,\theta}} e^{\frac{\lambda s}{t-s}} [e^\lambda - 1] \lambda^{-1-\gamma} \Big(\frac{\lambda}{t-s}\Big)^{\alpha+\gamma} B\Big(\Big(\frac{\lambda}{t-s}\Big)^{1+\alpha} I + B\Big)^{-1} w_1\, d\lambda, \end{aligned} \tag{34}$$

where the second equality follows by a change of variables. So

$$\|Bw(t) - Bw(s)\|_X \le |t-s|^\gamma \|w_1\|_{\mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha},\infty)} \frac{1}{2\pi} \int_{\Gamma_{0,\theta}} |1 - e^\lambda||\lambda|^{-1-\lambda}\, d|\lambda|. \quad (35)$$

By Lemma 5, and by the assumption on w_1, w is a strict solution. Therefore, $Bw(\underline{t}) \in \mathcal{C}([0,T];X)$. Hence the inequality (35) holds for $s = 0$ (with $Bw(0) = 0$). Thus $Bw(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$.

To see that (27) does hold with M independent of T, observe first that the factor on the right side of (35) multiplying $|t-s|^\gamma$ is independent of T. Then note that for $t \le 1$ one has, by (35), $\|Bw(t)\| \le c\|w_1\|_{\mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha},\infty)}$ and that for $t \ge 1$ there follows from (33),

$$\|Bw(t)\|_X \le \frac{1}{2\pi} \int_{\Gamma_{1,\theta}} |\frac{e^\lambda}{\lambda}|\, d|\lambda|\; \|w_1\|_{\mathcal{D}_B(\frac{\alpha}{1+\alpha},\infty)} \le M\|w_1\|_{\mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha},\infty)}\,. \quad (36)$$

Conversely, suppose that $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$, and that $Bw(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ for some $T > 0$. By Lemma 5, w is a strict solution, hence $Bw(0) = 0$. Also note that by (24), $\sup_{t\ge T}\|Bw(t)\|_X < \infty$. There follows, for $\lambda > 0$ and some constant $c > 0$,

$$\|\int_0^\infty e^{-\lambda t} Bw(t)\, dt\|_X \le c\int_0^\infty e^{-\lambda t} t^\gamma\, dt = c\Gamma(1+\gamma)\lambda^{-1-\gamma},$$

and so,

$$\sup_{\lambda>0} \|\lambda^{\alpha+\gamma} B(\lambda^{1+\alpha} I + B)^{-1} w_1\|_X < \infty\,;$$

thus $w_1 \in \mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha},\infty)$.

Next, let $1 < \gamma < 2$. The first part of the claim is now equivalent to

$$Bw_t(\underline{t}) \in \mathcal{C}^{\gamma-1}([0,T];X) \quad \text{iff} \quad Bw_1 \in \mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty).$$

First, suppose $w_1 \in \mathcal{D}(B)$ and $Bw_1 \in \mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)$. Then, for $t > 0$,

$$Bw_t(t) - Bw_1 = \frac{1}{2\pi i} \int_{\Gamma_{1,\theta}} e^{\lambda t}\{\lambda^\alpha(\lambda^{1+\alpha}I + B)^{-1}Bw_1 - \lambda^{-1}Bw_1\}\, d\lambda\,,$$

where we have used the fact that $\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{-1}\, d\lambda = 1$, for $r > 0$. Hence, using analyticity, for $r > 0$ and $0 < s < t$,

$$Bw_t(t) - Bw_t(s) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \{e^{\lambda s} - e^{\lambda t}\}\{\lambda^{-1}B(\lambda^{1+\alpha}I + B)^{-1}\}Bw_1\, d\lambda,$$

which gives, upon letting $r \to 0$ and changing variables,

$$Bw_t(t) - Bw_t(s) =$$

$$\frac{1}{2\pi i} \int_{\Gamma_{0,\theta}} e^{\frac{\lambda s}{t-s}}[1 - e^\lambda](\frac{\lambda}{t-s})^{-\gamma}\Big(\frac{\lambda}{t-s}\Big)^{\gamma-1} B\Big((\frac{\lambda}{t-s})^{1+\alpha}I + B\Big)^{-1} Bw_1 \frac{d\lambda}{t-s}, \quad (37)$$

and so,

$$\|Bw_t(t) - Bw_t(s)\|_X \le |t-s|^{\gamma-1}\|Bw_1\|_{\mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)}\frac{1}{2\pi}\int_{\Gamma_{0,\theta}}|1-\mathrm{e}^{\lambda}||\lambda|^{-\gamma}\,d|\lambda|\,. \quad (38)$$

Moreover, by the fact that w is a strict solution, we have $w_t \in \mathcal{C}([0,T];X)$, with $w_t(0) = w_1$. But B is closed, therefore (38) holds for $s = 0$. Hence, for $0 \le s < t$, and a constant c,

$$\|Bw_t(t) - Bw_t(s)\|_X \le c|t-s|^{\gamma-1}\|Bw_1\|_{\mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)}\,. \quad (39)$$

with $Bw_t(0) = Bw_1$. Thus $Bw_t(\underline{t}) \in \mathcal{C}^{\gamma-1}([0,T];X)$.

The first inequality in (36) (where $t \ge 1$) remains valid. From (39) and as $Bw(0) = 0$, there follows

$$\|Bw(t)\|_X \le c\|Bw_1\|_{\mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)}\,, \qquad t \in [0,1]. \quad (40)$$

Apply (39) for $t \in [0,1]$, and the relation

$$Bw_t - Bw_1 = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{r,\theta}}\mathrm{e}^{\lambda t}\lambda^{-1}B(\lambda^{1+\alpha}I+B)^{-1}Bw_1\,d\lambda$$

for $t \ge 1$, to obtain, for some constant c and $t \ge 0$,

$$\|Bw_t(t)\|_X \le c\|Bw_1\|_{\mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)}\,. \quad (41)$$

By (36), (39), (40), (41), the estimate (27) follows.

Conversely, suppose that $w_1 \in \mathcal{D}_B(\frac{\alpha}{1+\alpha})$, and that $Bw_t \in \mathcal{C}^{\gamma-1}([0,T];X)$ for some $T > 0$. By Lemma 5, $\sup_{t\ge T}\|Bw_t(t)\|_X < \infty$. Thus, for $\lambda > 0$ and some constant $c > 0$,

$$\left\|\int_0^\infty \mathrm{e}^{-\lambda t}B(w_t(t)-w_1)\,dt\right\|_X \le c\int_0^\infty \mathrm{e}^{-\lambda t}t^{\gamma-1}\,dt = c\Gamma(\gamma)\lambda^{-\gamma}.$$

Also,

$$\int_0^\infty \mathrm{e}^{-\lambda t}B(w_t(t)-w_1)\,dt = -\lambda^{-1}B(\lambda^{1+\alpha}I+B)^{-1}Bw_1, \quad \lambda > 0. \quad (42)$$

Thus,

$$\sup_{\lambda > o}\|\lambda^{\gamma-1}B(\lambda^{1+\alpha}I+B)^{-1}Bw_1\|_X < \infty,$$

i.e., $Bw_1 \in \mathcal{D}_B(\frac{\gamma-1}{1+\alpha},\infty)$.

(a) is proved.

(b) Assume that $B^{\frac{\alpha}{1+\alpha}}w_1 \in \mathcal{D}_B(\gamma,\infty)$.

First, we claim that there exists c_α such that for $\lambda \in \Gamma_{r,\theta}$, with $r > 0$,

$$\|B^{\frac{1}{1+\alpha}}(\lambda^{1+\alpha}I + B)^{-1}\|_{\mathcal{L}(X)} \le \frac{c_\alpha}{|\lambda|^\alpha}. \tag{43}$$

To see that (43) holds, note that if A is linear, densely defined and closed, with spectral angle $< \frac{\pi}{2}$, then (see, e.g., [11, (6.19), p. 73]) with $x \in \mathcal{D}(A)$ and $\beta \in (0,1)$, one has

$$\|A^\beta x\|_X \le c_\beta \|x\|_X^{1-\beta}\|Ax\|_X^\beta. \tag{44}$$

Let $y \in X$, take $A = B^{-1}$, $\beta = \frac{\alpha}{1+\alpha}$, $x = B(\lambda^{1+\alpha}I + B)^{-1}y$, and apply (44) and (2) to get

$$\begin{aligned}
&\|B^{\frac{1}{1+\alpha}}(\lambda^{1+\alpha}I + B)^{-1}y\|_X = \|B^{-\frac{\alpha}{1+\alpha}}B(\lambda^{1+\alpha}I + B)^{-1}y\|_X \\
\le& c_\alpha \|B(\lambda^{1+\alpha}I + B)^{-1}y\|_X^{\frac{1}{1+\alpha}} \|(\lambda^{1+\alpha}I + B)^{-1}y\|_X^{\frac{\alpha}{1+\alpha}} \\
\le& c_\alpha |\lambda|^{-\alpha}\|y\|_X,
\end{aligned}$$

and so (43) follows.

By (24), and after using analyticity to change the integration path,

$$Bw(t) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{t},\theta}} e^{\lambda t}\lambda^{-1+\alpha}B^{\frac{1}{1+\alpha}}(\lambda^{1+\alpha}I + B)^{-1}B^{\frac{\alpha}{1+\alpha}}w_1\, d\lambda, \quad t > 0,$$

or

$$\begin{aligned}
&\mu^\gamma B(\mu I + B)^{-1}Bw(t) = \\
&\frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{t},\theta}} \frac{e^{\lambda t}}{\lambda}\lambda^\alpha B^{\frac{1}{1+\alpha}}(\lambda^{1+\alpha}I + B)^{-1}\mu^\gamma B(\mu I + B)^{-1}B^{\frac{\alpha}{1+\alpha}}w_1\, d\lambda.
\end{aligned} \tag{45}$$

Hence, by (43) and by the assumption on $B^{\frac{\alpha}{1+\alpha}}w_1$,

$$\|\mu^\gamma B(\mu I + B)^{-1}Bw(t)\|_X \le c_\alpha \|B^{\frac{\alpha}{1+\alpha}}w_1\|_{\mathcal{D}_B(\gamma,\infty)}\frac{1}{2\pi}\int_{\Gamma_{\frac{1}{t},\theta}} |\frac{e^{\lambda t}}{\lambda}|\, d|\lambda|.$$

The last integral is bounded, independently of t. So $Bw(\underline{t}) \in \mathcal{B}(\mathbb{R}^+; \mathcal{D}(\gamma,\infty))$, and (28) holds.

Conversely, let $Bw(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T]; \mathcal{D}_B(\gamma,\infty))$. By Lemma 5, Bw is Laplace-transformable for $\lambda > 0$. Moreover,

$$w_1 = \lambda^2\int_0^\infty e^{-\lambda t}w(t)\, dt + \lambda^{1-\alpha}\int_0^\infty e^{-\lambda t}Bw(t)\, dt, \quad \lambda > 0.$$

Thus (recall (ii)), for $\lambda, \mu > 0$,

$$\begin{aligned}
&\mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}}B^2(\mu I + B^2)^{-1}w_1 = \lambda^2\int_0^\infty e^{-\lambda t}\mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}}B^2(\mu I + B^2)^{-1}w(t)\, dt \\
&+\lambda^{1-\alpha}\int_0^\infty e^{-\lambda t}\mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}}B^2(\mu I + B^2)^{-1}Bw(t)\, dt.
\end{aligned} \tag{46}$$

Let $\mu > 0$ be arbitrary and fix $\lambda = \mu^{\frac{1}{2(1+\alpha)}}$. We split the integrands on the right in parts; over $[0, T]$ and $[T, \infty)$, respectively. For the first part of the first integral we have

$$\Big\|\lambda^2 \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} w(t)\, dt\Big\|_X =$$
$$\Big\|\lambda^2 \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\alpha}{2(1+\alpha)}-\frac{1}{2}} \mu^{\frac{\gamma}{2}+\frac{1}{2}} B^2(\mu I + B^2)^{-1} w(t)\, dt\Big\|_X \leq$$
$$c\lambda \mu^{-\frac{1}{2(1+\alpha)}} \|w(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_{B^2}(\frac{\gamma}{2}+\frac{1}{2},\infty))} \leq c\|Bw(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))},$$

by the choice of λ and the mapping properties of B. For the first part of the second integral there follows

$$\Big\|\lambda^{1-\alpha} \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\alpha}{2(1+\alpha)}} \mu^{\frac{\gamma}{2}} B^2(\mu I + B^2)^{-1} Bw(t)\, dt\Big\|_X \leq$$
$$\Big(\lambda^{1-\alpha} \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\alpha}{2(1+\alpha)}}\, dt\Big) \|Bw(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_{B^2}(\frac{\gamma}{2},\infty))} \leq$$
$$c\lambda^{-\alpha} \mu^{\frac{\alpha}{2(1+\alpha)}} \|Bw(\underline{t}))\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} = c\|Bw(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))},$$

by the choice of λ.

For the second part of the first integral one obtains

$$\Big\|\lambda^2 \int_T^\infty \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}-\frac{1}{2}} \mu^{\frac{1}{2}} B^2(\mu I + B^2)^{-1} w(t)\, dt\Big\|_X$$
$$\leq c\big(\lambda \mathrm{e}^{-\lambda T} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}-\frac{1}{2}}\big) \|Bw(\underline{t})\|_{\mathcal{B}(\mathbb{R}^+;X)} = c\mu^{\frac{\gamma}{2}} \mathrm{e}^{-T\mu^{\frac{1}{2(1+\alpha)}}} \|Bw(\underline{t})\|_{\mathcal{B}(\mathbb{R}^+;X)}.$$

For the second part of the second integral one gets

$$\|\lambda^{1-\alpha} \int_T^\infty \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} Bw(t)\, dt\|_X$$
$$\leq c\lambda^{-\alpha} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} \mathrm{e}^{-\lambda T} \|Bw(\underline{t})\|_{\mathcal{B}(\mathbb{R}^+;X)} = c\mu^{\frac{\gamma}{2}} \mathrm{e}^{-T\mu^{\frac{1}{2(1+\alpha)}}} \|Bw(\underline{t})\|_{\mathcal{B}(\mathbb{R}^+;X)},$$

where we have used $\sup_{\mu>0}\|B^2(\mu I + B^2)^{-1}\|_{\mathcal{L}(X)} < \infty$.

Collecting the four estimates, we conclude that

$$\sup_{\mu>0}\|\mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} w_1\|_X < \infty.$$

Thus $w_1 \in \mathcal{D}_{B^2}(\frac{\gamma}{2} + \frac{\alpha}{2(1+\alpha)}, \infty)$ and so $B^{\frac{\alpha}{1+\alpha}} w_1 \in \mathcal{D}_B(\gamma, \infty)$.

(b) is proved.

(c) First, let $\gamma \in (0, 1)$.

Suppose $w_1 \in \mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha})$, and let $\epsilon > 0$ be arbitrary. Then choose $\lambda_\epsilon > 0$ such that

$$\frac{1}{2\pi} \int_{\Gamma_{0,\theta} \cap |\lambda| \leq \lambda_\epsilon} |1 - \mathrm{e}^{\lambda}||\lambda|^{-1-\gamma}\, d|\lambda| < \epsilon, \tag{47}$$

and $h_\epsilon > 0$ such that

$$\Big\|\Big(\frac{\lambda}{t-s}\Big)^{\alpha+\gamma}B\Big(\big(\frac{\lambda}{t-s}\big)^{1+\alpha}I+B\Big)^{-1}w_1\Big\|_X < \epsilon, \tag{48}$$

uniformly for $\lambda \in \Gamma_{0,\theta}$, with $|\lambda| \geq \lambda_\epsilon$, $0 < t-s < h_\epsilon$. Then, by (34), (35), (47), (48),

$$\|Bw(t)-Bw(s)\|_X \leq \epsilon|t-s|^\gamma \|w_1\|_{\mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha})} + \epsilon|t-s|^\gamma \frac{1}{2\pi}\int_{\Gamma_{0,\theta}} |1-\mathrm{e}^\lambda||\lambda|^{-1-\gamma}\, d|\lambda|.$$

Finally recall the continuity of $Bw(s)$ at $s=0$. Thus $Bw(\underline{t}) \in h^\gamma([0,T];X)$, for $T>0$ arbitrary.

Conversely, suppose $Bw(\underline{t}) \in h^\gamma([0,T];X)$ for some $T>0$. By Lemma 5, $Bw(\underline{t}) \in \mathcal{B}(\mathbb{R}^+;X)$. Then, for $\lambda > 0$,

$$\Big\|\int_0^\infty \mathrm{e}^{-\lambda t}Bw(t)\,dt\Big\|_X \leq \int_0^\infty c(t)\mathrm{e}^{-\lambda t}t^\gamma\, dt = \lambda^{-1-\gamma}\int_0^\infty c(\frac{s}{\lambda})\mathrm{e}^{-s}s^\gamma\, ds,$$

where $c(t)$ is bounded and continuous on $\mathbb{R}^+$, with $c(0)=0$. Consequently,

$$\lim_{\lambda\to\infty}\int_0^\infty c(\frac{s}{\lambda})\mathrm{e}^{-s}s^\gamma\, ds = 0,$$

and so, recall that $B\tilde{w} = \lambda^{\alpha-1}B(\lambda^{1+\alpha}I+B)^{-1}w_1$,

$$\lim_{\lambda\to\infty}\|\lambda^{\alpha+\gamma}B(\lambda^{1+\alpha}I+B)^{-1}w_1\|_X = 0,$$

which implies that $w_1 \in \mathcal{D}_B(\frac{\alpha+\gamma}{1+\alpha})$.

Next, let $1<\gamma<2$, and assume that $w_1 \in \mathcal{D}_{B^2}\big(\frac{\alpha+\gamma}{2(1+\alpha)}\big)$, or, equivalently, that $Bw_1 \in \mathcal{D}_B(\frac{\gamma-1}{1+\alpha})$. Then recall (37), fix $\epsilon>0$, take $\lambda_\epsilon>0$ such that

$$\frac{1}{2\pi}\int_{\Gamma_{0,\theta}\cap|\lambda|\leq\lambda_\epsilon} |1-\mathrm{e}^\lambda||\lambda|^{-\gamma}\, d|\lambda| < \epsilon, \tag{49}$$

and $h_\epsilon > 0$ such that

$$\Big\|\big(\frac{\lambda}{t-s}\big)^{\gamma-1}B\Big(\big(\frac{\lambda}{t-s}\big)^{1+\alpha}I+B\Big)^{-1}Bw_1\Big\|_X \leq \epsilon, \tag{50}$$

uniformly for $\lambda \in \Gamma_{0,\theta}$ with $|\lambda| \geq \lambda_\epsilon$, $0<t-s<h_\epsilon$. Upon combining (37), (49), (50), one then obtains $Bw_t(\underline{t}) \in h^{\gamma-1}([0,T];X)$ for any $T>0$.

Conversely, let $Bw_t(\underline{t}) \in h^{\gamma-1}([0,T];X)$ for some $T>0$. Then, for $\lambda>0$,

$$\Big\|\int_0^\infty \mathrm{e}^{-\lambda t}[B(w_t(t)-w_1)]\,dt\Big\|_X \leq \int_0^\infty c(t)\mathrm{e}^{-\lambda t}t^{\gamma-1}\, dt = \lambda^{-\gamma}\int_0^\infty c(\frac{s}{\lambda})\mathrm{e}^{-s}s^{\gamma-1}\, ds\ ,$$

where $c(t)$ is bounded, continuous on $\mathbb{R}^+$, with $c(0) = 0$. Therefore, $\lim_{\lambda\to\infty}\int_0^\infty c(\frac{s}{\lambda})e^{-s}s^{\gamma-1}\,ds = 0$. So, recall (42),

$$\lim_{\lambda\to\infty}\left\|\lambda^{\gamma-1}B(\lambda^{1+\alpha}I+B)^{-1}Bw_1\right\|_X = 0,$$

and $Bw_1 \in \mathcal{D}_B(\frac{\gamma-1}{1+\alpha})$ follows. Equivalently, one has $w_1 \in \mathcal{D}_{B^2}\big(\frac{\alpha+\gamma}{2(1+\alpha)}\big)$.

Thus (c) is proved.

(d) Let

$$B^{\frac{\alpha}{1+\alpha}}w_1 \in \mathcal{D}_B(\gamma). \tag{51}$$

By the proof of (b),

$$Bw(\underline{t}) \in \mathcal{C}(\mathbb{R}^+;X)\cap\mathcal{B}(\mathbb{R}^+;\mathcal{D}_B(\gamma,\infty)).$$

By (43), (45), one has $Bw(t) \in \mathcal{D}_B(\gamma)$ for each $t > 0$ and $\lim_{\lambda\to\infty}\|\lambda^\gamma B(\lambda I + B)^{-1}Bw(\tau)\|_X = 0$, uniformly in τ. To get continuity, write (λ_0 to be chosen)

$$\begin{aligned}&\|Bw(t)-Bw(s)\|_{\mathcal{D}_B(\gamma)} = \sup_{\lambda>0}\|\lambda^\gamma B(\lambda I+B)^{-1}(Bw(t)-Bw(s))\|_X \le\\ &2\sup\|\lambda^\gamma B(\lambda I+B)^{-1}Bw(\tau)\|_X + \sup_{0<\lambda\le\lambda_0}\|\lambda^\gamma B(\lambda I+B)^{-1}(Bw(t)-Bw(s))\|_X,\end{aligned}$$

where the second sup is taken over $0<\lambda_0\le\lambda\,,\tau\in\mathbb{R}^+$. Fix $\epsilon>0$. Then choose λ_0 large enough so that for any $\tau\ge 0$,

$$2\sup_{\lambda_0\le\lambda}\|\lambda^\gamma B(\lambda I+B)^{-1}Bw(\tau)\|_X \le 2^{-1}\epsilon.$$

Let $T>0$ be arbitrary, and choose $t,s\in[0,T]$ such that $|t-s|$ is small enough, to get (by the fact that $Bw(\underline{t})\in\mathcal{C}(\mathbb{R}^+;X)$),

$$\begin{aligned}&\sup_{0<\lambda\le\lambda_0}\|\lambda^\gamma B(\lambda I+B)^{-1}(Bw(t)-Bw(s))\|_X \le\\ &\lambda_0^\gamma\sup_{\lambda>0}\|B(\lambda I+B)^{-1}\|_{\mathcal{L}(X)}\|Bw(t)-Bw(s)\|_X \le \frac{\epsilon}{2}.\end{aligned}$$

Thus

$$Bw(\underline{t}) \in \mathcal{C}\big([0,T];\mathcal{D}_B(\gamma)\big), \tag{52}$$

for any $T>0$.

Conversely, assume (52). We claim that (51) follows. For this it suffices to show that if $\epsilon>0$ is arbitrary, then there exists μ_ϵ such that if $\mu\ge\mu_\epsilon$, then

$$\|\mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}}B^2(\mu I+B^2)^{-1}w_1\|_X \le \epsilon.$$

By (52), and by the mapping properties of B,

$$w(\underline{t}) \in \mathcal{C}\Big([0,T];\mathcal{D}_{B^2}\big(\frac{\gamma}{2}+\frac{1}{2}\big)\Big),$$

$$Bw(\underline{t}) \in \mathcal{C}\Big([0,T]; \mathcal{D}_{B^2}(\frac{\gamma}{2})\Big).$$

Thus there exists μ_ϵ such that if $\mu \geq \mu_\epsilon$, then

$$\sup_{0\leq t\leq T} \|\mu^{\frac{\gamma+1}{2}} B^2(\mu I + B^2)^{-1} w(t)\|_X \leq \frac{\epsilon}{4}\,,$$

$$\sup_{0\leq t\leq T} \|\mu^{\frac{\gamma}{2}} B^2(\mu I + B^2)^{-1} Bw(t)\|_X \leq \frac{\epsilon}{4}\,.$$

Then, c.f. the proof of (b), if $\mu \geq \mu_\epsilon$,

$$\Big\|\lambda^2 \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} w(t)\, dt\Big\|_X \leq \frac{\epsilon}{4}\lambda\mu^{-\frac{1}{2(1+\alpha)}} = \frac{\epsilon}{4}\,, \tag{53}$$

provided λ is chosen so that $\lambda\mu^{-\frac{1}{2(1+\alpha)}} = 1$.

Analogously,

$$\Big\|\lambda^{1-\alpha} \int_0^T \mathrm{e}^{-\lambda t} \mu^{\frac{\alpha}{2(1+\alpha)}} \mu^{\frac{\gamma}{2}} B^2(\mu I + B^2)^{-1} Bw(t)\, dt\Big\|_X \leq \frac{\epsilon}{4}\,, \tag{54}$$

by the same choice of λ.

Finally, observe that without loss of generality we may take μ large enough so that (c.f. the proof of (b))

$$\|\lambda^2 \int_T^\infty \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} w(t)\, dt\|_X \leq \frac{\epsilon}{4}, \tag{55}$$

$$\|\lambda^{1-\alpha} \int_T^\infty \mathrm{e}^{-\lambda t} \mu^{\frac{\gamma}{2}+\frac{\alpha}{2(1+\alpha)}} B^2(\mu I + B^2)^{-1} Bw(t)\, dt\|_X \leq \frac{\epsilon}{4}. \tag{56}$$

Upon combining (53), (54), (55), (56) with (46) we obtain $w_1 \in \mathcal{D}_{B^2}(\frac{\gamma}{2} + \frac{\alpha}{2(1+\alpha)})$, hence (51).

This completes the proof of Lemma 6.

6. **An interpolation Lemma.**

For the analysis of necessary conditions for solutions of (1) to exhibit a specific behavior one needs the following version of earlier interpolation results [2, Lemma 3], [9, Prop. 2.2.12].

Below, $\mathcal{C}^1(I,X)$ denotes Lipschitz-continuous functions defined on I with values in X and $\mathcal{C}^2$ denotes functions having the first derivative in $\mathcal{C}^1$.

Lemma 10. *Let X and Y be Banach spaces that are continuously injected in a Hausdorff locally convex topological vector space. Let I be a closed, bounded interval and let $f \in \mathcal{C}^{\tilde\alpha}(I,X) \cap \mathcal{C}^{\tilde\beta}(I,Y)$, where $\tilde\alpha, \tilde\beta \in (0,2]$. Then $f \in \mathcal{C}^{(1-\theta)\tilde\alpha+\theta\tilde\beta}(I,(X,Y)_{\theta,\infty})$ for each $\theta \in (0,1)$ and*

$$\|f\|_{\mathcal{C}^{(1-\theta)\tilde\alpha+\theta\tilde\beta}(I;(X,Y)_{\theta,\infty})} \leq 2\big(\|f\|_{\mathcal{C}^{\tilde\alpha}(I;X)} + \|f\|_{\mathcal{C}^{\tilde\beta}(I;Y)}\big) + \begin{cases} 2^{\tilde\alpha}|I|^{1-\tilde\alpha} \sup_{t\in I}\|f'(t)\|_X, & \textit{if } \tilde\alpha > 1,\ \tilde\beta \leq 1, \\ 2^{\tilde\beta}|I|^{1-\tilde\beta} \sup_{t\in I}\|f'(t)\|_Y, & \textit{if } \tilde\beta > 1,\ \tilde\alpha \leq 1. \end{cases} \tag{57}$$

Moreover, if $\tilde{\alpha} \neq \tilde{\beta}$*, and* $(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}=1$*, then* $f' \in \mathcal{B}(I,(X,Y)_{\theta,\infty})$ *and* (57) *holds with* $\|f'\|_{\mathcal{B}(I,(X,Y)_{\theta,\infty})}$ *on the left side of the inequality.*

If $f \in h^{\tilde{\alpha}}(I,X) \cap h^{\tilde{\beta}}(I,Y)$*, then* $f \in h^{(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}}(I,(X,Y)_\theta)$ *for each* $\theta \in (0,1)$*. Moreover, if* θ *is such that* $(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}=1$*, and* $\tilde{\alpha} \neq \tilde{\beta}$*, then* $f' \in \mathcal{C}(I,(X,Y)_\theta)$*.*

Proof of Lemma 10. For the relation (57), see [2, Lemma 3]. The fact that (57) holds with $\|f'\|_{\mathcal{B}(I,(X,Y)_{\theta,\infty})}$ on the left side (under the stated assumptions on $\tilde{\alpha}$, $\tilde{\beta}$) is contained in the proof of [2, Lemma 3].

Let $f \in h^{\tilde{\alpha}}(I,X) \cap h^{\tilde{\beta}}(I,Y)$. Then choose $f_n \in \mathcal{C}^\infty(I,X\cap Y)$ such that $f_n \to f$ in $\mathcal{C}^{\tilde{\alpha}}(I,X) \cap \mathcal{C}^{\tilde{\beta}}(I,Y)$. By (57), $f_n \to f$ in $\mathcal{C}^{(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}}(I,(X,Y)_{\theta,\infty})$, and if $\tilde{\alpha} \neq \tilde{\beta}$, $(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}=1$, then $f_n' \to f'$ in $\mathcal{B}(I,(X,Y)_{\theta,\infty})$. Moreover, $f_n \in h^{(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}}(I,(X,Y)_\theta)$ and $f_n' \in \mathcal{C}(I,(X,Y)_\theta)$. But these last spaces are closed in $\mathcal{C}^{(1-\theta)\tilde{\alpha}+\theta\tilde{\beta}}(I,(X,Y)_{\theta,\infty})$ and $\mathcal{C}(I,(X,Y)_{\theta,\infty})$, respectively, and so the last part of the Lemma follows. □

Let us briefly indicate an immediate use of the above Lemma for the first step of the converse analysis. We only consider case (a), and take $f=0$.

By (1), and assuming $Bu \in \mathcal{C}^\gamma([0,T];X)$, $\gamma \in (0,1)$, one has $D_t^\alpha(u_t - u_1) \in \mathcal{C}^\gamma([0,T];X)$. Then $u \in \mathcal{C}^{1+\alpha}([0,T];X)$. Apply Lemma 10 with $\tilde{\alpha}=1+\alpha$, $\tilde{\beta}=\gamma$, $\theta=\frac{\alpha}{1+\alpha-\gamma} \in (0,1)$, $Y=\mathcal{D}(B)$ and $f=u$. This gives $u_1=u'(0) \in \mathcal{D}_B(\frac{\alpha}{1+\alpha-\gamma},\infty)$. Then, by (a) of Lemma 6, $Bw \in \mathcal{C}^{\frac{\alpha\gamma}{1+\alpha-\gamma}}$. By the relation $Bu = Bv + Bw$ one has $Bv \in \mathcal{C}^{\frac{\alpha\gamma}{1+\alpha-\gamma}}$. Thus, by (a) of Lemma 4, we arrive at $Bu_0 \in \mathcal{D}_B(\frac{\gamma\alpha}{(1+\alpha-\gamma)(1+\alpha)},\infty)$.

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(continued from the back cover)

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