

# SMOOTHNESS IN FRACTIONAL EVOLUTION EQUATIONS AND CONSERVATION LAWS

G. Gripenberg Ph. Clément S-O. Londen



TEKNILLINEN KORKEAKOULU  
TEKNISKA HÖGSKOLAN  
HELSINKI UNIVERSITY OF TECHNOLOGY  
TECHNISCHE UNIVERSITÄT HELSINKI  
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

**G. Gripenberg, Ph. Clément, and S-O. Londen:** Smoothness in fractional evolution equations and conservation laws; Helsinki University of Technology Institute of Mathematics Research Reports A419 (1999).

**Abstract:** *The regularity of solutions of the equation*

$$(D_t^\alpha(u - u_0))(t, x) + \sigma(u)_x(t, x) = f(t, x), \quad t, x \geq 0,$$

where  $D_t^\alpha$  denotes the fractional derivative, is studied in the case where  $\sigma' > 0$ . It is also shown that the solution to the Riemann problem for the fractional Burgers equation (where  $\sigma(r) = \frac{1}{2}r^2$ ) is continuous and has compact support (in the  $x$ -direction). A result on the continuity of the interface is established. In order to prove these results it is first shown that if  $A$  is an  $m$ -accretive operator in a Banach space,  $k$  is log-convex with  $\lim_{t \downarrow 0} k(t) = +\infty$ , and if  $u$  is the solution of

$$\frac{d}{dt} \int_0^t k(t-s)(u(s) - y) ds + A(u(t)) \ni f(t), \quad t > 0, \quad u(0) = y,$$

then  $A(u(t))$  is continuous when  $t > 0$ .

**AMS subject classifications:** 35K99, 35L99, 45K05.

**Keywords:** fractional derivative, evolution equation, Hölder continuity, conservation law.

ISBN 951-22-4661-9

ISSN 0784-3143

Libella Painopalvelu Oy, Espoo, 1999

Helsinki University of Technology  
Department of Engineering Physics and Mathematics  
Institute of Mathematics  
P.O. Box 1100, 02015 HUT, Finland  
email: [math@hut.fi](mailto:math@hut.fi)  
downloadables: <http://www.math.hut.fi/>

author's email: [gustaf.gripenberg@hut.fi](mailto:gustaf.gripenberg@hut.fi)

# 1 Introduction

Recently a new type of approximation of scalar conservation laws in several variables has been introduced in [3]. Rather than adding a viscosity term (for this approach see, e.g., [8]), the order of derivation with respect to time is lowered, that is, the derivative is replaced by a fractional derivative of order  $\alpha \in (0, 1)$ . Furthermore, instead of using the Crandall-Liggett theorem as is done in [4], an abstract result, [10], is employed to establish the existence of a *strong* solution. In [3] the convergence of these strong solutions as  $\alpha \uparrow 1$  to the entropy solution of  $u_t + \operatorname{div} \mathbf{g}(u) = 0$  is proven and some estimates for the speed of convergence are established.

The aim of this paper is to investigate further these solutions in the one-dimensional case, i.e., we analyze the regularity of solutions of the nonlinear fractional conservation law

$$D_t^\alpha(u - u_0) + \sigma(u)_x = f. \quad (1)$$

Here  $D_t^\alpha$  denotes the fractional derivative of order  $\alpha \in (0, 1)$ , see [14, p. 133], i.e.,

$$(D_t^\alpha v)(t) \stackrel{\text{def}}{=} \frac{d}{dt} \int_0^t g_{1-\alpha}(t-s)v(s) ds, \quad t > 0,$$

$$(D_t^\alpha v)(0) \stackrel{\text{def}}{=} \lim_{h \downarrow 0} \frac{1}{h} \int_0^h g_{1-\alpha}(h-s)v(s) ds,$$

where

$$g_\beta(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\beta)} t^{\beta-1}, \quad t > 0, \quad \beta > 0,$$

and where  $v$  is (at least) continuous and satisfies  $v(0) = 0$ .

As an important tool for studying this equation we consider the abstract fractional nonlinear evolution equation

$$\frac{d}{dt} \int_0^t k(t-s)(u(s) - y) ds + A(u(t)) \ni f(t), \quad t > 0, \quad u(0) = y. \quad (2)$$

In (2),  $u$  is the unknown function with range in a Banach space  $X$ ,  $y \in X$  and  $f : \mathbb{R}^+ \rightarrow X$  are given, and  $k$  is a locally integrable real-valued function with a singularity at the origin. The nonlinear operator  $A$  may be multivalued and maps  $\mathcal{D}(A) \subset X$  into (subsets of)  $X$ . Our primary current interest concerns the continuity and boundedness of the function  $A(u(\underline{t}))$ .

In [10], the existence of a strong solution  $u$ , satisfying  $A(u) \in L_{\text{loc}}^1(\mathbb{R}^+; X)$ , was obtained. Conditions implying that the solution  $u$  is continuous were given in [3].

In this paper we demonstrate that under rather weak hypotheses one has  $A(u) \in \mathcal{C}((0, \infty); X)$ . In addition this function is uniformly bounded on  $(0, T]$  for each  $T > 0$ . Subsequently, these facts are applied to examine the regularity of the solution of (1).

As a first application we get the continuity of the solution of the Riemann problem for the fractional Burgers equation, i.e., for equation (1) with  $\sigma(\underline{u}) = \frac{1}{2}\underline{u}^2$  and  $f = 0$ . This improves on a result of [11] concerning (1). In this result it is assumed that  $\sigma'(\underline{u}) \geq c_0 > 0$ . The special structure of the fractional Burgers equation implies that the solution vanishes when  $x \geq \Gamma(1 - \alpha)t^\alpha$ , in contrast to the linear case where there is an infinite speed of propagation. We also establish a result on the continuity of the interface.

Furthermore, in Theorem 3 the results obtained on (2) are combined with earlier Schauder estimates on linear equations, [2], to establish results on the smoothness of solutions of (1).

The regularity, both temporal and spatial, of solutions of equations involving fractional derivatives of order  $\alpha \in (1, 2)$  have been studied in several papers; [5], [6], and [7]. See also the monograph [12] for further results and references.

## 2 Statement of results

Our result on (2) is the following.

**Theorem 1.** *Assume that  $X$  is a real Banach space and that*

- (i)  $k \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$  is positive and nonincreasing,  $\lim_{t \downarrow 0} k(t) = +\infty$ , and  $\log(k(\underline{t}))$  is convex;
- (ii)  $A$  is an  $m$ -accretive operator on  $X$ ;
- (iii)  $y \in \hat{\mathcal{D}}(A)$ , i.e.,  $y \in X$  and  $\sup_{\lambda > 0} \|A_\lambda y\|_X < \infty$ ;
- (iv)  $f \in \mathcal{C}(\mathbb{R}^+; X)$  is such that  $\int_0^T \omega_{f,T}(s) |k'(s)| ds < \infty$  for each  $T > 0$  where  $\omega_{f,T}$  is the modulus of continuity of  $f$ , i.e.,  $\omega_{f,T}(\underline{s}) \stackrel{\text{def}}{=} \sup_{t_1, t_2 \in [0, T], |t_1 - t_2| \leq \underline{s}} \|f(t_1) - f(t_2)\|_X$ .

Then there is a unique strong solution  $u$  of (2) such that  $u \in \mathcal{C}(\mathbb{R}^+; X)$ ,  $u(0) = y$ , and there is a function  $w \in \mathcal{C}((0, \infty); X)$  such that  $\sup_{0 < t < T} \|w(t)\|_X < \infty$  for each  $T > 0$ ,  $w(t) \in A(u(t))$  for all  $t > 0$  and

$$\frac{d}{dt} \int_0^t k(t-s)(u(s) - y) ds + w(t) = f(t), \quad t > 0. \quad (3)$$

Moreover, if  $0 \leq t < t+h \leq \tau$  then

$$\begin{aligned} \|u(t+h) - u(t)\|_X &\leq \int_0^t \|f(t+h-s) - f(t-s)\|_X r(s) ds \\ &\quad + \left( \sup_{\tau \in [0, h]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right) \int_t^{t+h} r(s) ds, \end{aligned} \quad (4)$$

where  $r$  satisfies

$$\int_{[0, t]} k(t-s)r(s) ds = 1, \quad t \in (0, \tau]. \quad (5)$$

A function  $u : \mathbb{R}^+ \rightarrow X$  is a strong solution of (2) if there exists a function  $w \in L^1_{\text{loc}}(\mathbb{R}^+; X)$  such that  $w(\underline{t}) \in A(u(\underline{t}))$  a.e. on  $\mathbb{R}^+$  and  $\int_0^t k(t-s)(u(s) - y) ds = \int_0^t (f(s) - w(s)) ds$  for every  $t \geq 0$ .

Our next result concerns the homogeneous version of (1) with, essentially  $\sigma(\underline{r}) = c\underline{r}^\gamma$ ,  $\gamma > 1$ . In particular, this includes the fractional Burgers equation.

**Theorem 2.** *Assume that*

- (i)  $k \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$  is positive and nonincreasing,  $\lim_{t \downarrow 0} k(t) = +\infty$ , and  $\log(k(\underline{t}))$  is convex;
- (ii)  $\sigma \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$  is strictly increasing on  $(0, 1)$  and there are constants  $C$  and  $\gamma > 1$  such that

$$\frac{1}{C}r^\gamma \leq \sigma(r) \leq Cr^\gamma, \quad r \in [0, 1].$$

Then there is a solution  $u$  of the Riemann problem

$$\frac{d}{dt} \int_0^t k(t-s)(u(s, x) - \chi_{(-\infty, 0]}(x)) ds + \sigma(u)_x(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad (6)$$

$$u(0, x) = \chi_{(-\infty, 0]}(x), \quad x \in \mathbb{R},$$

which is continuous for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{(0, 0)\}$  and is such that for each  $t > 0$  the function  $x \rightarrow u(t, x)$  is absolutely continuous and nonincreasing, for each  $x \in \mathbb{R}$  the function  $t \mapsto u(t, x)$  is nondecreasing (so that the function  $t \mapsto \int_0^t k(t-s)(u(s, x) - \chi_{(-\infty, 0]}(x)) ds$  is locally absolutely continuous), and equation (6) holds a.e. on  $\mathbb{R}^+ \times \mathbb{R}$ . Moreover,

$$u(t, x) = 0 \text{ when } x \geq \frac{1}{k(t)} \int_0^1 \frac{\sigma'(r)}{r} dr, \quad t > 0, \quad (7)$$

and the function

$$\varphi(\underline{t}) \stackrel{\text{def}}{=} \inf\{x > 0 \mid u(\underline{t}, x) = 0\}$$

is continuous and strictly increasing.

Let  $X$  be a (complex) Banach space and let  $I$  be an interval. The Hölder spaces  $\mathcal{C}^{(\gamma)}(I; X)$ ,  $\gamma \in [0, 1]$ , are defined by

$$\mathcal{C}^{(\gamma)}(I; X) \stackrel{\text{def}}{=} \left\{ f : I \rightarrow X \mid \sup_{\substack{s, t \in I \\ s \neq t}} \frac{\|f(t) - f(s)\|_X}{|t - s|^\gamma} < \infty \right\},$$

with norm

$$\|f\|_{\mathcal{C}^{(\gamma)}(I)} \stackrel{\text{def}}{=} \sup_{t \in I} \|f(t)\|_X + \sup_{\substack{s, t \in I \\ s \neq t}} \frac{\|f(t) - f(s)\|_X}{|t - s|^\gamma}.$$

If  $\gamma \in (1, 2]$ , then  $\mathcal{C}^{(\gamma)}(I; X) \stackrel{\text{def}}{=} \{f \in \mathcal{C}^1(I; X) \mid f' \in \mathcal{C}^{(\gamma-1)}(I; X)\}$  with norm  $\|f\|_{\mathcal{C}^{(\gamma)}(I)} \stackrel{\text{def}}{=} \sup_{t \in I} \|f(t)\|_X + \|f'\|_{\mathcal{C}^{(\gamma-1)}(I)}$ . Observe that  $\mathcal{C}^{(0)} \neq \mathcal{C}$  and  $\mathcal{C}^{(1)} \neq \mathcal{C}^1$ .

We consider a function of two variables to be a function of the first variable with values in a function space, that is,  $f(\underline{t}, \underline{x})$  is the function  $t \mapsto (x \mapsto f(t, x))$ .

**Theorem 3.** *Assume that  $\alpha \in (0, 1)$ ,  $\tau > 0$ ,  $\xi > 0$ ,  $\mu \in (0, \alpha)$ , and that*

- (i)  $\sigma \in \mathcal{C}_{\text{loc}}^{(2)}(\mathbb{R}; \mathbb{R})$  and  $\sigma'(\underline{x}) > 0$ ;
- (ii)  $u_0 \in \mathcal{C}^{(1+\frac{\mu}{\alpha})}([0, \xi]; \mathbb{R})$  and  $u_0(0) = u_0'(0) = 0$ ;
- (iii)  $f \in \mathcal{C}^{(\mu)}([0, \tau], \mathcal{C}([0, \xi]; \mathbb{R})) \cap \mathcal{C}^{(\alpha+\delta)}([0, \tau], L^1([0, \xi]; \mathbb{R}))$  where  $\delta > 0$ , and  $f(\underline{t}, 0) = 0$  and  $f(0, \underline{x}) \in \mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi]; \mathbb{R})$ .

*Then there is a unique solution  $u$  of (1) on  $(0, \tau] \times (0, \xi]$  with  $u(\underline{t}, 0) = 0$  and  $u(0, \underline{x}) = u_0(\underline{x})$  such that  $u_x \in \mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}([0, \xi]; \mathbb{R}))$ .*

### 3 Proofs

*Proof of Theorem 1.* Let  $\{k_n\}_{n=1}^\infty$  be a sequence of functions that satisfy the assumption (i), except that  $\lim_{t \downarrow 0} k_n(t) < \infty$ , and are such that  $\lim_{n \rightarrow \infty} \int_0^t k_n(s) ds = \int_0^t k(s) ds$ ,  $\lim_{n \rightarrow \infty} k_n(t) = k(t)$ ,  $\lim_{n \rightarrow \infty} k_n'(t) = k'(t)$ , and  $|k_n'(t)| \leq |k'(t)|$  for all  $t > 0$ . We let  $\rho_n$  be the first kind resolvent associated with  $k_n$  (cf. (5)); then  $\rho_n$  has the pointmass  $1/k_n(0)$  at 0 and is otherwise induced by an integrable function, that is

$$\rho_n([0, \underline{t}]) = \frac{1}{k_n(0)} + \int_0^{\underline{t}} r_n(s) ds, \quad t \geq 0,$$

where  $r_n$  is nonnegative and nonincreasing, because  $k_n$  is log-convex, see [9, Lemma 2.1]. When  $k$  is replaced by  $k_n$  one can use (ii) and a standard fixed-point argument to show that there is a unique solution of (2); we denote this solution by  $u_n$ . It is a consequence of [3, Thm. 1] that  $u_n$  converges uniformly on compact subsets of  $\mathbb{R}^+$  to a continuous function  $u$ . However, we need to know more. In particular our next purpose is to show that  $w \in \mathcal{C}((0, \infty); X)$  where  $w(\underline{t}) \in A(u(\underline{t}))$  is defined by (14).

By [3, formula (24)] we have for  $0 \leq t < t + h$

$$\begin{aligned} \|u_n(t+h) - u_n(t)\|_X &\leq \int_{[0, t]} \|f(t+h-s) - f(t-s)\|_X \rho_n(ds) \\ &+ \left( \sup_{\tau \in [0, h]} \|f(\tau)\|_X + \|A_{1/k_n(0)}(y)\|_X \right) \\ &\times \int_{[0, t]} \left( \int_{[0, h]} (k_n(t-s) - k_n(t-s+h-\sigma)) \rho_n(d\sigma) \right) \rho_n(ds). \end{aligned} \tag{8}$$

Now a straightforward calculation using (5) (with  $k$  and  $r$  replaced by  $k_n$  and  $\rho_n$ , respectively) shows that

$$\begin{aligned} \int_{[0,t]} \left( \int_{[0,h]} (k_n(t-s) - k_n(t-s+h-\sigma)) \rho_n(d\sigma) \right) \rho_n(ds) \\ = \rho_n((t, t+h]) = \int_t^{t+h} r_n(s) ds. \end{aligned} \quad (9)$$

By [3, Thm. 1], (8), (9), and by the fact that  $\lim_{n \rightarrow \infty} \rho_n([0, t]) = \int_0^t r(s) ds$ , we get (4).

By a change of variables,

$$\int_0^h \left( \int_{t-s}^t r_n(\sigma) d\sigma \right) |k'_n|(s) ds = \int_{t-h}^t (k_n(t-\sigma) - k_n(h)) r_n(\sigma) d\sigma, \quad 0 < h \leq t.$$

Since the functions  $r_n$  are nonincreasing, it follows that

$$\lim_{h \downarrow 0} \int_0^h \left( \int_{t-s}^t r_n(\sigma) d\sigma \right) |k'_n|(s) ds = 0, \quad (10)$$

uniformly for  $n \geq 1$  and uniformly for  $t$  in a compact subset of  $(0, \infty)$ . Since  $|k'_n(\underline{t})| \leq |k'(\underline{t})|$  we deduce from (iv) that

$$\lim_{h \downarrow 0} \int_0^h \omega_{f,\tau}(s) |k'_n(s)| ds = 0 \text{ uniformly in } n. \quad (11)$$

Use (9) in (8), replace  $t+h$  and  $t$  by  $t$  and  $t-s$ , respectively, multiply by  $|k'_n(s)|$ , integrate with respect to  $s$  over  $[0, h]$  and let  $h \downarrow 0$ . This gives, by (10) and (11),

$$\lim_{h \downarrow 0} \int_0^h \|u_n(t-s) - u_n(t)\|_X |k'_n(s)| ds = 0, \quad (12)$$

uniformly for  $n \geq 1$  and uniformly for  $t$  in a compact subset of  $(0, \infty)$ .

Now we can rewrite (2) (with  $k$  replaced by  $k_n$ ) for each  $t \geq 0$  as

$$k_n(t)(u_n(t) - y) + \int_0^t (u_n(t-s) - u_n(t)) k'_n(s) ds + A(u_n(t)) \ni f(t). \quad (13)$$

By (12), and as  $u_n$  converges uniformly on compact subsets of  $\mathbb{R}^+$  to the continuous function  $u$ , it follows that  $k_n(t)(u_n(t) - y) + \int_0^t (u_n(t-s) - u_n(t)) k'_n(s) ds$  converges uniformly on each compact subset of  $(0, \infty)$  to  $k(t)(u(t) - y) + \int_0^t (u(t-s) - u(t)) k'(s) ds$  which must then be a continuous function on  $(0, \infty)$ . Let

$$w(t) \stackrel{\text{def}}{=} f(t) - k(t)(u(t) - y) - \int_0^t (u(t-s) - u(t)) k'(s) ds, \quad (14)$$

so that (3) holds with  $w \in \mathcal{C}((0, \infty), X)$ . Since  $A$  is  $m$ -accretive it is also closed and therefore we have by (13) and by the convergence results that  $w(t) \in A(u(t))$  for all  $t > 0$ .

It remains to show that  $w$  is bounded on  $(0, T]$  for each  $T > 0$ . Since  $u(0) = y$  we get from (4), when we take  $t = 0$ , that

$$\|u(h) - y\|_X \leq \left( \sup_{\tau \in [0, h]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right) \int_0^h r(s) ds, \quad h > 0.$$

Because  $k$  is nonincreasing there follows by (5) that  $k(\underline{t}) \int_0^{\underline{t}} r(s) ds \leq 1$  so that

$$\|k(t)(u(t) - y)\|_X \leq \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right).$$

Similarly, replace  $t$  and  $t + h$  in (4) by  $t - s$  and  $t$ , respectively, multiply by  $|k'(s)|$  and integrate over  $[0, t]$  to obtain

$$\begin{aligned} \left\| \int_0^t (u(t-s) - u(t)) k'(s) ds \right\| &\leq \int_0^t \omega_{f, \tau}(s) \int_0^{t-s} r(\sigma) d\sigma |k'(s)| ds \\ &+ \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right) \int_0^t \left( \int_{t-s}^t r(\sigma) d\sigma \right) |k'(s)| ds. \end{aligned}$$

Moreover, by (5),

$$\int_0^t \left( \int_{t-s}^t r(\sigma) d\sigma \right) |k'(s)| ds = \int_0^t (k(t-\sigma) - k(t)) r(\sigma) d\sigma \leq 1,$$

and so by the fact that  $k$  and  $r$  are nonnegative and by (iii) and (iv) we get the desired conclusion.  $\square$

*Proof of Theorem 2.* Since we will show that the solution takes its values in the interval  $[0, 1]$  we may without loss of generality assume that  $\sigma \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$  is strictly increasing on  $\mathbb{R}$ .

We easily see that by taking  $u(t, x) = 1$  for  $x \leq 0$  and  $t \geq 0$  we have a solution in that region and we are left with the equation

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^t k(t-s) u(s, x) ds + \sigma(u)_x(t, x) &= 0, \quad t > 0, \quad x > 0, \\ u(t, 0) &= 1, \quad t > 0, \\ u(0, x) &= 0, \quad x > 0, \end{aligned} \tag{15}$$

In [11, Lemma 3] it is shown that if one lets  $\mathcal{D}(A) = \{ u \in L^1(\mathbb{R}^+; \mathbb{R}) \mid \sigma(u) \in AC(\mathbb{R}^+; \mathbb{R}), u(0) = 1, \sigma(u)' \in L^1(\mathbb{R}^+; \mathbb{R}) \}$ , and defines  $A(u) = \sigma(u)'$ ,  $u \in \mathcal{D}(A)$ , then  $A$  is a closed,  $m$ -accretive operator in  $L^1(\mathbb{R}^+; \mathbb{R})$ . By [11, Theorem 5] there exists a solution  $u$  of (15), which is nonincreasing in the  $x$ -variable and nondecreasing in the  $t$ -variable, such that the function  $x \mapsto \sigma(u(t, x))$  is absolutely continuous for almost every  $t > 0$ , and such that the



function  $t \mapsto \int_0^t k(t-s)u(s, x) ds$  is locally absolutely continuous for every  $x \geq 0$ , and (15) holds almost everywhere.

By Theorem 1 we know that the function  $t \mapsto \sigma(u(t, \underline{x}))_x \in L^1(\mathbb{R}^+; \mathbb{R})$  is continuous on  $(0, \infty)$  and that (15) holds in  $L^1(\mathbb{R}^+; \mathbb{R})$  for all  $t > 0$ . Since  $\sigma(u(t, 0)) = \sigma(1)$  for all  $t > 0$  and  $\sigma(u(t, x)) = \int_0^x \sigma(u(t, y))_x dy + \sigma(u(t, 0))$  it follows that  $\sigma(u)$  is continuous in  $(0, \infty) \times \mathbb{R}^+$  and since  $\sigma$  is strictly increasing the same result holds for  $u$ . By Theorem 1 we also know that  $u(t, \underline{x}) \rightarrow 0$  in  $L^1(\mathbb{R}^+; \mathbb{R})$  as  $t \downarrow 0$  and from the monotonicity properties of  $u$  we can therefore conclude that  $u$  is continuous in  $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0, 0)\}$ .

Next we derive an inequality that we will use repeatedly below. Assume that  $x_0 \stackrel{\text{def}}{=} \varphi(t_0) < \infty$  and that  $x_0 < x_1 \leq \varphi(t_1)$  where  $t_1 > t_0 \geq 0$ . From the proof of Theorem 1 we know that for each  $t > 0$  we have

$$\frac{\partial}{\partial t} \int_0^t k(t-s)u(s, x) ds \stackrel{\text{a.e.}}{=} k(t)u(t, x) + \int_0^t (u(t-s, x) - u(t, x))k'(s) ds, \quad x > 0,$$

(where the derivative with respect to  $t$  is a function with values in  $L^1(\mathbb{R}^+; \mathbb{R})$ ). Since  $u(s, x) = 0$  when  $s \leq t_0$  and  $x > x_0$  (by the monotonicity properties of  $u$ ), we can rewrite this equality for  $t > t_0$  as

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^t k(t-s)u(s, x) ds \\ \stackrel{\text{a.e.}}{=} k(t-t_0)u(t, x) + \int_0^{t-t_0} (u(t-s, x) - u(t, x))k'(s) ds, \quad x > x_0. \end{aligned}$$

Because  $k$  is nonincreasing and  $u$  is nondecreasing in its first variable, it follows from the fact that (1) (or equivalently (3)) holds that for each  $t > t_0$  we get

$$k(t-t_0)u(t, x) + \sigma'(u(t, x))u_x(t, x) \stackrel{\text{a.e.}}{\leq} 0, \quad x > x_0.$$

In particular, if we choose  $t = t_1$ , then we know that  $u(t, x) > 0$  for  $x_0 < x < x_1$  and it follows by the continuity of  $u$  that

$$k(t_1 - t_0)(x_1 - x_0) \leq \int_{u(t_1, x_1)}^{u(t_1, x_0)} \frac{\sigma'(r)}{r} dr. \quad (16)$$

Since clearly  $\varphi(0) = 0$  we may take  $t_0 = 0$ . Because the function  $\frac{\sigma'(\underline{x})}{\underline{x}}$  is integrable on  $[0, 1]$  and  $k(\underline{t}) > 0$ , we see from (16) that we have  $\varphi(t_1) < \infty$  and that (7) holds.

The monotonicity properties of  $u$  imply that  $\varphi$  is nondecreasing. By the continuity of the function  $u$  it follows that  $\varphi$  is continuous from the left, so in order to establish the claim about continuity we suppose to the contrary that there is a point  $t_0 \geq 0$  such that  $\lim_{t \downarrow t_0} \varphi(t) = \varphi(t_0) + \delta$  for some  $\delta > 0$ . If we choose  $x_0 \stackrel{\text{def}}{=} \varphi(t_0)$  and  $x_1 = x_0 + \delta$ , then  $x_1 \leq \varphi(t_1)$  for each  $t_1 > t_0$  and

we get a contradiction from (16) if we let  $t_1 \downarrow t_0$ . Thus we have established the continuity of  $\varphi$ .

It remains to prove that  $\varphi$  is strictly increasing. Suppose that this is not the case but that there are two points  $t_1 < t_2$  such that  $\varphi(t_1) = \varphi(t_2)$ . By the continuity of  $u$  we know that  $t_1 > 0$  and that we can choose  $t_1$  such that  $\varphi(t) < \varphi(t_1)$  when  $0 \leq t < t_1$ . We define  $x_1 = \varphi(t_1)$ .

We shall derive a contradiction and first we show that

$$\lim_{x \uparrow x_1} \sigma(u(t_1, x))(x_1 - x)^{-\frac{\gamma}{\gamma-1}} = \infty. \quad (17)$$

Write  $\frac{\sigma'(r)}{r} = \frac{\sigma(r)}{r^2} + \frac{d}{dr}\left(\frac{\sigma(r)}{r}\right)$ , use the inequalities in (ii), and the facts that  $\gamma > 1$  and  $\sigma(u(t_1, x_1)) \geq 0$ , to conclude from (16) that when  $0 < t_0 < t_1$  and  $x_0 = \varphi(t_0)$  we have

$$k(t_1 - t_0)(x_1 - x_0) \leq \frac{\gamma}{\gamma-1} C^{\frac{2\gamma-1}{\gamma}} \sigma(u(t_1, x_0))^{\frac{\gamma-1}{\gamma}}.$$

Since  $\varphi(t) < \varphi(t_1)$  when  $0 \leq t < t_1$  it follows that  $t_0 \uparrow t_1$ , and hence  $k(t_1 - t_0) \uparrow \infty$ , when  $x_0 \uparrow x_1$ . By the above inequality we therefore obtain (17).

Next, let  $y$  be some small positive number and integrate both sides of equation (15) over  $(x_1 - y, x_1)$ . Then we get, because  $u(t, x_1) = 0$  for all  $t \in (0, t_2]$ ,

$$\frac{d}{dt} \int_0^t k(t-s) \int_{x_1-y}^{x_1} u(s, v) dv ds = \sigma(u(t, x_1 - y)), \quad t \in (0, t_2]. \quad (18)$$

We let  $r$  be the resolvent of first kind of  $k$ , that is,  $r$  satisfies (5). Our assumptions on  $k$  guarantee that such a resolvent exists and that it is positive and nonincreasing, see [9, Lemma 2.1]. Take the convolution (with respect to  $t$ ) of both sides of (18) with the function  $p(\underline{t}) \stackrel{\text{def}}{=} \int_0^{\underline{t}} r(\underline{t}-s) s^\alpha ds$  where  $\alpha > \frac{2-\gamma}{\gamma-1}$ . By (5),

$$\int_0^{t_2} (t_2 - s)^\alpha \int_{x_1-y}^{x_1} u(s, v) dv ds = \int_0^{t_2} p(t_2 - s) \sigma(u(s, x_1 - y)) ds. \quad (19)$$

Using Hölder's inequality twice to estimate the left hand side of (19), we obtain

$$\begin{aligned} & \int_0^{t_2} (t_2 - s)^\alpha \int_{x_1-y}^{x_1} u(s, v) dv ds \\ & \leq \int_0^{t_2} (t_2 - s)^\alpha \left( \int_{x_1-y}^{x_1} u(s, v)^\gamma dv \right)^{\frac{1}{\gamma}} ds y^{\frac{\gamma-1}{\gamma}} \\ & \leq \left( \int_0^{t_2} p(t_2 - s) \int_{x_1-y}^{x_1} u(s, v)^\gamma dv ds \right)^{\frac{1}{\gamma}} y^{\frac{\gamma-1}{\gamma}} \left( \int_0^{t_2} \frac{s^{\frac{\alpha\gamma}{\gamma-1}}}{p(s)^{\frac{1}{\gamma-1}}} ds \right)^{\frac{\gamma-1}{\gamma}}. \end{aligned} \quad (20)$$

Since  $r$  is nonincreasing and not identically zero there exists a constant  $c_1$  such that  $p(t) \geq c_1 t^{\alpha+1}$  when  $t \in [0, t_2]$  and therefore it follows from our choice of  $\alpha$  that

$$\int_0^{t_2} \frac{s^{\frac{\alpha\gamma}{\gamma-1}}}{p(s)^{\frac{1}{\gamma-1}}} ds < \infty. \quad (21)$$

If we now let

$$w(y) \stackrel{\text{def}}{=} \int_0^{t_2} p(t_2 - s) \int_{x_1 - y}^{x_1} \sigma(u(s, v)) dv ds,$$

then the right hand side of (19) equals  $w'(y)$ , and so by (ii), (20), and by (21) there is a constant  $c_2$  such that

$$w'(y) \leq c_2 y^{\frac{\gamma-1}{\gamma}} w(y)^{\frac{1}{\gamma}}.$$

Since  $w(0) = 0$  and  $w(y) > 0$  for  $y > 0$  we get

$$w(y) \leq \left( c_2 \frac{\gamma-1}{2\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} y^{\frac{2\gamma-1}{\gamma-1}},$$

and we conclude that there is a constant  $c_3$  such that

$$w'(y) \leq c_3 y^{\frac{\gamma}{\gamma-1}}. \quad (22)$$

But from the definition of  $w$ , from the fact that  $u$  is nondecreasing in its first variable, and by the monotonicity of  $\sigma$  it follows that

$$w'(y) \geq \int_0^{t_2 - t_1} p(s) ds \sigma(u(t_1, x_1 - y)).$$

When this inequality is combined with (17) (where we take  $x = x_1 - y$ ) and (22), a contradiction follows. This completes the proof.  $\square$

*Proof of Theorem 3.* The idea of the proof is roughly as follows: First we show that if one has a solution for  $t$  on some interval  $[0, T]$  (one clearly has such a solution when  $T = 0$ ), then it can be extended to a slightly larger interval. From the proof of this fact one sees that if this extension procedure does not give a solution on the entire interval  $[0, \tau]$  then there is some maximal interval  $[0, \hat{\tau})$  on which there is a solution and which is such that  $\sup_{T < \hat{\tau}} \|\sigma'(u)\|_{C^{(\mu)}([0, T]; C([0, \xi]))} = \infty$ . In order to show that this last fact leads to a contradiction we then apply the same argument as when establishing the existence of a local solution, but we derive estimates for  $\|u_x\|_{C^{(\mu)}([0, T]; L^1([0, \mathfrak{X}]))}$  instead of estimating  $\|u_x\|_{C^{(\mu)}([0, T]; C([0, \mathfrak{X}]))}$ . It is of crucial importance for this part of the proof that we derive these estimates for all  $\mathfrak{X} \in [0, \xi]$ . In this connection, the use of Theorem 1 is essential.

First we show that we may, without loss of generality, assume that there are positive constants  $c_0$ ,  $c_1$ , and  $c_2$  such that

$$0 < c_0 \leq \sigma'(\underline{r}) \leq c_1 < \infty \text{ and } \sup_{r \neq s} \frac{|\sigma'(r) - \sigma'(s)|}{|r - s|} \leq c_2 < \infty. \quad (23)$$

By (i) it is sufficient to show that there is an apriori bound for the solution. In analogy with the proof of Theorem 2 we let

$$\mathcal{D}(A) = \{ v \in L^1([0, \xi]; \mathbb{R}) \mid \sigma(v) \in AC([0, \xi]; \mathbb{R}), v(0) = 0 \}, \quad (24)$$

and

$$A(v) = \sigma(v)', \quad v \in \mathcal{D}(A). \quad (25)$$

Then one can easily show (cf. the proof of [11, Lemma 3]) that  $A$  is a closed,  $m$ -accretive operator in  $L^1([0, \xi]; \mathbb{R})$  and that  $\|(I + \lambda A)^{-1}v\|_{L^\infty([0, \xi])} \leq \|v\|_{L^\infty([0, \xi])}$  for all  $v \in L^\infty([0, \xi]; \mathbb{R})$  when  $\lambda > 0$ . Then it follows from [3, Thm. 4.(a), Prop. 5] that if we find a solution  $u$  of (1), then it must satisfy  $\sup_{x \in [0, \xi]} |u(t, x)| \leq \sup_{x \in [0, \xi]} |u_0(x)| + \int_0^t g_\alpha(t-s) \sup_{x \in [0, \xi]} |f(s, x)| ds$  and this is the desired apriori bound. Thus we shall for the rest of the proof assume that (23) holds.

Suppose next that there is a number  $T \in [0, \tau)$  such that there is a solution  $u \in \mathcal{C}([0, T] \times [0, \xi]; \mathbb{R})$  of (1) on  $(0, T] \times (0, \xi]$  such that  $u_x \in \mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]; \mathbb{R}))$ ,  $u(0, \underline{x}) = u_0(\underline{x})$  and  $u(\underline{t}, 0) = 0$ ; if  $T = 0$  this solution is taken to be  $u(0, \underline{x}) = u_0(\underline{x})$  (so that this hypothesis holds at least with  $T = 0$ ).

We intend to show that this solution can be continued to  $[0, \hat{T}] \times [0, \xi]$  where  $\hat{T} > T$  and  $\hat{T} - T$  is sufficiently small. We do this in two steps. In the first step we solve (27) with  $c$  given; in the second step we find a fixed-point for the map  $c \mapsto \sigma'(v)$  (where  $v$  is the solution of (27) obtained in the first step). This continuation procedure is concluded by formula (41).

Thus we first show (using the same argument as in the proof of [2, Thm. 1]) that there are constants  $\delta$  and  $M_1$  depending on  $\alpha, \mu, \tau, \xi, c_0$ , and  $c_1$  such that if  $\hat{T} \in (T, \tau]$  and  $c \in \mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{R}))$  satisfy

$$\begin{aligned} c_0 &\leq c(\underline{t}, \underline{x}) \leq c_1, \\ c(t, x) &= \sigma'(u(t, x)), \quad (t, x) \in [0, T] \times [0, \xi], \\ (\hat{T} - T)^\mu \|c\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} &\leq \delta, \end{aligned} \quad (26)$$

then there exists a unique solution  $v$  of the equation

$$\begin{aligned} (D_t^\alpha(v - u_0))(t, x) + c(t, x)v_x(t, x) &= f(t, x), \quad (t, x) \in (0, \hat{T}] \times (0, \xi], \\ v(0, x) &= u_0(x), \quad x \in [0, \xi], \\ v(t, 0) &= 0, \quad t \in [0, \hat{T}], \end{aligned} \quad (27)$$

such that (clearly  $v(t, x) = u(t, x)$  for  $(t, x) \in [0, T] \times [0, \xi]$ )

$$\begin{aligned} \|v_x\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} &\leq M_1 \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} \\ &+ M_1 \|f\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}([0, \xi]))} + M_1 \|\sigma'(u_0(\underline{x}))u_0'(\underline{x}) - f(0, \underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])} \\ &+ M_1 \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} \|c\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}, \quad \mathfrak{X} \in [0, \xi]. \end{aligned} \quad (28)$$

Observe that the first and last term of this inequality are written in terms of the space variable  $\underline{x} \in [0, \xi]$ . The proof of the existence of  $v$  satisfying (27) will be completed by the paragraph containing formula (39).

To solve (27), we begin by studying the following equation:

$$(D_t^\alpha(v - u_0))(t, x) + b(x)v_x(t, x) = g(t, x), \quad t \in (0, \tau], \quad x \in (0, \xi], \quad (29)$$

with boundary condition  $v(\underline{t}, 0) = 0$  and initial condition  $v(0, \underline{x}) = u_0(\underline{x})$  under the following assumption on the function  $b$ :

$$b \in \mathcal{C}(\mathbb{R}^+; \mathbb{R}) \text{ and } 0 < c_0 \leq b(\underline{x}) \leq c_1 < \infty. \quad (30)$$

We denote by  $B_b$  the linear operator in  $\mathcal{C}_{0 \rightarrow 0}([0, \xi]; \mathbb{C}) \stackrel{\text{def}}{=} \{q \in \mathcal{C}([0, \xi]; \mathbb{C}) \mid q(0) = 0\}$  with domain

$$\mathcal{D}(B_b) = \{q \in \mathcal{C}^1([0, \xi]; \mathbb{C}) \mid q(0) = q'(0) = 0\}$$

and defined by

$$(B_b q)(x) = b(x)q'(x), \quad x \in [0, \xi], \quad q \in \mathcal{D}(B_b).$$

We denote by  $B$  the corresponding operator with  $b(\underline{x}) = 1$  and  $\xi$  replaced by  $\xi_0 = \xi/c_0$ .

Thus (29) can be written as

$$D_t^\alpha(v - u_0) + B_b v = g. \quad (31)$$

Next, perform a change of variable  $y = \int_0^x \frac{1}{b(s)} ds$ , so that equation (31) is replaced by

$$D_t^\alpha(v^b - u_0^b) + B v^b = g^b, \quad (32)$$

where

$$\begin{aligned} g^b(\underline{t}, \underline{y}) &= g(\underline{t}, \rho(\underline{y})), & y \in [0, \xi_b] \\ u_0^b(\underline{y}) &= u_0(\rho(\underline{y})), \end{aligned}$$

and

$$\begin{aligned} g^b(\underline{t}, \underline{y}) &= g(\underline{t}, \xi), \\ u_0^b(\underline{y}) &= u_0(\xi) + b(\xi)u_0'(\xi)(y - \xi_b), & y \in (\xi_b, \xi_0]. \end{aligned}$$

Here  $\xi_b = \int_0^\xi \frac{1}{b(s)} ds$  and  $\rho$  is the inverse of the function  $x \mapsto \int_0^x \frac{1}{b(s)} ds$ . By [1, Thm. 6.(a)] equation (32) has a unique solution  $v^b$  which satisfies the bound

$$\begin{aligned} &\|Bv^b(\underline{t}) - g^b(0)\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}_{0 \rightarrow 0}([0, \xi_0])}) \\ &\leq M_2 \left( \|Bu_0^b - g^b(0)\|_{\mathcal{C}^{(\frac{\mu}{c_0})}([0, \xi_0])} + \|g^b(\underline{t}) - g^b(0)\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}_{0 \rightarrow 0}([0, \xi_0])}) \right), \end{aligned}$$

where  $M_2$  depends on  $\alpha$ ,  $\mu$ ,  $\tau$  and  $\xi_0$ . Now we change variables back again, that is, we define

$$v(\underline{t}, x) = v^b \left( \underline{t}, \int_0^x \frac{1}{b(s)} ds \right), \quad \text{for } x \in [0, \xi]. \quad (33)$$

We can therefore conclude that there is a unique solution  $v$  of (29) such that

$$\|v_x\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}([0, \xi]; \mathbb{C}))} \leq M_3 \left( \|b(\underline{x})u'_0(\underline{x}) - g(0, \underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])} + \|g\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}([0, \xi]))} \right), \quad (34)$$

where (with some crude estimates)  $M_3 = \frac{1}{c_0}(M_2 \max\{2, c_1^{\frac{\mu}{\alpha}}\} + 1)$ .

Our next claim is that (34) holds with  $\tau$  replaced by an arbitrary  $\hat{T} \in (0, \tau]$ ,  $\xi$  replaced by an arbitrary  $\mathfrak{X} \in [0, \xi]$ , and with  $M_3$  unchanged. To see this, choose  $\hat{T} \in (0, \tau]$ ,  $\mathfrak{X} \in [0, \xi]$ , and redefine  $b$ ,  $u_0$  and  $g$  as  $b(x) = b(\mathfrak{X})$ ,  $u_0(x) = u_0(\mathfrak{X}) + u'_0(\mathfrak{X})(x - \mathfrak{X})$ , and  $g(t, x) = g(t, \mathfrak{X})$  for  $x \in (\mathfrak{X}, \xi]$  and  $t \in [0, \hat{T}]$  and  $g(t, x) = g(\hat{T}, x)$  for  $x \in [0, \xi]$  and  $t \in (\hat{T}, \tau]$ . Then we can use the uniqueness of the solution and the definition of the Hölder norms to conclude that we in fact have our claim, i.e.,

$$\|v_x\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \mathfrak{X}]; \mathbb{C}))} \leq M_3 \left( \|b(\underline{x})u'_0(\underline{x}) - g(0, \underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \mathfrak{X}])} + \|g\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \mathfrak{X}]))} \right), \quad \hat{T} \in (0, \tau], \quad \mathfrak{X} \in [0, \xi]. \quad (35)$$

Choose

$$\delta = \frac{1}{4M_3}, \quad (36)$$

and  $\hat{T} \in (T, \tau]$  such that the last part of (26) holds. Having a solution of (29) satisfying (35) and having chosen  $\hat{T}$ , we proceed to find a solution of (27).

Let  $P$  denote the set

$$P \stackrel{\text{def}}{=} \{p \in \mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{C})) \mid p(t, \underline{x}) = u_x(t, \underline{x}), \quad 0 \leq t \leq T\}.$$

For each  $p \in P$  we have to find a solution  $w$  of the equation

$$D_t^\alpha(w - u_0)(\underline{t}, \underline{x}) + c(T, \underline{x})w_x(\underline{t}, \underline{x}) = f(\underline{t}, \underline{x}) + (c(T, \underline{x}) - c(\underline{t}, \underline{x}))p(\underline{t}, \underline{x}), \quad (37)$$

on  $[0, \hat{T}] \times [0, \xi]$  with boundary condition  $w(\underline{t}, 0) = 0$  (and initial condition  $w(0, \underline{x}) = u_0(\underline{x})$ ) and  $c$  as in (26). Note that this equation is of type (29). Observe also that the right-hand side of (37) evaluated at  $t = 0$  is

$$f(0, x) + (c(T, x) - c(0, x))u'_0(x),$$

and therefore the term  $b(x)u'_0(\underline{x}) - g(0, \underline{x})$  appearing in (35) is now, when  $b(\underline{x}) = c(T, \underline{x})$ , equal to  $c(0, x)u'_0(x) - f(0, x)$ . Thus we conclude from (ii) and from the results above on (29) that we can find a solution  $w$  of (37) such that  $w_x \in \mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{C}))$ . Moreover, the uniqueness guarantees that we have  $w_x \in P$ .

Let us denote the mapping  $p \rightarrow w_x$  by  $w_x = G(p)$ . Using the linearity of equation (37), and (35) with  $b(\underline{x}) = c(T, \underline{x})$  once more, we conclude that

$$\begin{aligned} & \| (G(p_1) - G(p_2))(t, \underline{x}) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} \\ & \leq M_3 \| (c(T, \underline{x}) - c(t, \underline{x}))(p_1 - p_2)(t, \underline{x}) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}, \quad \mathfrak{X} \in [0, \xi]. \end{aligned} \quad (38)$$

Let  $p_\Delta = p_1 - p_2$  and  $c_\Delta(t, \underline{x}) = c(T, \underline{x}) - c(t, \underline{x})$ . Since  $p_1$  and  $p_2 \in P$  it follows that  $p_\Delta(t, \underline{x}) = 0$  for  $t \in [0, T]$  and therefore we can, when analyzing the term  $(c(T, \underline{x}) - c(t, \underline{x}))(p_1 - p_2)(t, \underline{x})$ , assume that  $c(t, \underline{x}) = c(T, \underline{x})$  for  $t \in [0, T]$ . Thus we conclude from the last part of (26) and from (36) that

$$\sup_{\substack{t \in [0, \hat{T}] \\ x \in [0, \mathfrak{X}]}} |c_\Delta(t, x)p_\Delta(t, x)| \leq \frac{1}{4M_3} \sup_{\substack{t \in [0, \hat{T}] \\ x \in [0, \mathfrak{X}]}} |p_\Delta(t, x)|, \quad \mathfrak{X} \in [0, \xi].$$

Furthermore, if we write  $c_\Delta(t, x)p_\Delta(t, x) - c_\Delta(s, x)p_\Delta(s, x) = c_\Delta(t, x)(p_\Delta(t, x) - p_\Delta(s, x)) + (c_\Delta(t, x) - c_\Delta(s, x))(p_\Delta(s, x) - p_\Delta(T, x))$  using the fact that  $p_\Delta(T, \underline{x}) = 0$ , and use (26) once again, then we conclude that

$$\begin{aligned} & \| (c(T, x) - c(t, x))(p_1 - p_2)(t, x) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} \\ & \leq \frac{1}{2M_3} \| (p_1 - p_2)(t, x) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}, \quad \mathfrak{X} \in [0, \xi]. \end{aligned}$$

Hence we have, using (38), for every  $\mathfrak{X} \in [0, \xi]$ ,

$$\begin{aligned} & \| (G(p_1) - G(p_2))(t, x) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} \\ & \leq \frac{1}{2} \| (p_1 - p_2)(t, x) \|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}, \end{aligned} \quad (39)$$

and we see that the mapping  $G$  is a contraction and that there is a unique fixed-point, i.e., a function  $v$  such that  $v_x = G(v_x)$ . Thus we get a solution of (27) on the interval  $[0, \hat{T}]$ .

If we take  $p_0 \in P$  to be such that  $p_0(t, \underline{x}) = u_x(T, \underline{x})$  for  $t \in [T, \hat{T}]$  then  $\|p_0\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))} = \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))}$ . Using inequality (35) to estimate  $\|G(p_0)\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}$  and then (39) to estimate  $\|G(v_x) - G(p_0)\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}$ , we conclude that (28) holds with

$$M_1 = \max\{1 + 4M_3 \|c\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))}, 2M_3\}.$$

With  $c$  fixed, the solution  $v$  of (27) can of course be continued to  $[0, \tau] \times [0, \xi]$ . However, our goal is to solve (1), i.e., (27) with  $c(t, \underline{x}) = \sigma'(v(t, \underline{x}))$ . For this purpose we apply another fixed-point argument on  $[0, \hat{T}]$  with  $\hat{T} - T$  sufficiently small (and recall that we have a solution of (1) on  $[0, T]$ ).

We let  $M_4$  be the constant

$$M_4 \stackrel{\text{def}}{=} c_1 + c_2 \max\{1, \xi\} M_1 \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} \\ + \xi c_2 M_1 \|f\|_{\mathcal{C}^{(\mu)}([0, \tau]; \mathcal{C}([0, \xi]))} + \xi c_2 M_1 \|\sigma'(u_0(\underline{x}))u'_0(\underline{x}) - f(0, \underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])},$$

and choose  $\hat{T} \in (T, \tau]$  such that

$$(\hat{T} - T)^\mu \leq \frac{\delta}{M_4 e^{M_4 \xi}}. \quad (40)$$

For our fixed-point argument we let

$$V = \left\{ c \in \mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{R})) \mid c(t, x) = \sigma'(u(t, x)), \quad t \in [0, T], \quad x \in [0, \xi], \right. \\ \left. c_0 \leq c(t, x) \leq c_1, \quad t \in [T, \hat{T}], \quad x \in [0, \xi], \right. \\ \left. \|c\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \mathfrak{X}]))} \leq M_4 e^{M_4 \mathfrak{X}}, \quad \mathfrak{X} \in [0, \xi] \right\}.$$

Note that  $V$  is convex and not empty. Now we define the function  $F(c)$  for  $c \in V$  by

$$F(c)(\underline{t}, \underline{x}) \stackrel{\text{def}}{=} \sigma'(v(\underline{t}, \underline{x})),$$

where  $v$  is the solution of (27). (By the definition of  $V$  and by (40) condition (26) is satisfied and hence such a (unique) solution exists.)

By the uniqueness we know that we have  $F(c)(t, x) = \sigma'(u(t, x))$  for  $t \in [0, T]$  and  $x \in [0, \xi]$  and by (23) we also have  $c_0 \leq F(c)(\underline{t}, \underline{x}) \leq c_1$ . Finally we note that since  $v(\underline{t}, 0) = 0$  we have

$$F(c)(\underline{t}, \underline{x}) = \sigma' \left( \int_0^{\underline{x}} v_x(\underline{t}, r) \, dr \right),$$

and it follows that

$$\|F(c)\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \mathfrak{X}]))} \leq c_1 + c_2 \int_0^{\mathfrak{X}} \|v_x\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, r]))} \, dr \\ \leq M_4 + M_4 \int_0^{\mathfrak{X}} \|c\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, r]))} \, dr \leq M_4 e^{M_4 \mathfrak{X}}, \quad \mathfrak{X} \in [0, \xi],$$

where the second inequality is a consequence of (28) and the definition of  $M_4$ , and where the last inequality follows because  $c \in V$ . This shows that  $F(c) \in V$ .

Finally we observe that by [2, Thm. 1 and (4)] the set of solutions of (27) one gets when  $c \in V$  is contained in a bounded subset of  $\mathcal{C}^{((\mu+\alpha)/2)}([0, \hat{T}]; \mathcal{C}^{(1/2)}([0, \xi]; \mathbb{R}))$  (for example) and therefore this set of solutions, and hence also  $F(c) = \sigma'(v)$  for  $c \in V$  is contained in a compact subset of  $\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{R}))$ . (Note in particular that since our boundary condition is now  $v(\underline{t}, 0) = 0$  we do not need the assumption that the function  $x \mapsto c(\underline{t}, x)$  is a continuous function



with values in  $\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathbb{R})$ . Therefore the constant  $M$  appearing in [2, formula (4)] depends on  $\|c\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]))}$ ,  $c_0$  and  $c_1$ , but not otherwise on  $c$ .) Thus we know by the Schauder fixed-point theorem that there is a function  $c \in V$  such that  $F(c) = c$  and the corresponding solution of (27) is then the unique solution of (1) on  $[0, \hat{T}] \times [0, \xi]$ .

If the claim of the theorem does not hold there is, by the continuation argument above, a maximal number  $\hat{\tau} \in (0, \tau]$  such that there is a solution of (1) on  $(0, \hat{\tau}) \times (0, \xi]$ , and such that  $u_x \in \mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]; \mathbb{R}))$  for all  $T \in (0, \hat{\tau})$ . If  $\sup_{T < \hat{\tau}} \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} < \infty$ , then this solution can be continued by the argument used above, and we get a contradiction. Furthermore, it also follows from the argument in the above that if  $\sup_{T < \hat{\tau}} \|\sigma'(u)\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} < \infty$ , then  $\sup_{T < \hat{\tau}} \|u_x\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} < \infty$ . Thus we assume that

$$\sup_{T < \hat{\tau}} \|\sigma'(u)\|_{\mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]))} = \infty, \quad (41)$$

and we will derive a contradiction from this.

We want to apply Theorem 1 and therefore we define the operator  $A$  by (24) and (25). It is straightforward to check that by (ii)  $y = u_0$  belongs to  $\mathcal{D}(A) \subset \hat{\mathcal{D}}(A)$  and that by (iii) the function  $t \mapsto f(t, \underline{x}) \in L^1([0, \xi]; \mathbb{R})$  satisfies the assumption (iv) of Theorem 1. Thus Theorem 1 may be applied to (1) and so we obtain the existence of a unique (strong) solution  $u \in \mathcal{C}([0, \tau]; L^1([0, \xi]; \mathbb{R}))$ . By uniqueness, this solution coincides with the one constructed above on  $[0, \hat{\tau}) \times [0, \xi]$ .

It follows from Theorem 1, together with the results on the local solution that we already have established, that the function

$$t \mapsto \sigma(u)_x(t, \underline{x}) \in L^1([0, \xi]; \mathbb{R}) \text{ is uniformly continuous on } [0, \hat{\tau}).$$

An immediate consequence of this result, of (23), and of the fact that  $u(\underline{t}, 0) = 0$ , is that

$$u \text{ is uniformly continuous on } [0, \hat{\tau}) \times [0, \xi], \quad (42)$$

and hence we also conclude that

$$t \mapsto u_x(t, \underline{x}) \in L^1([0, \xi]; \mathbb{R}) \text{ is uniformly continuous on } [0, \hat{\tau}). \quad (43)$$

In the above, the results of [1] were applied to the operator  $u \mapsto u_x$  in the space of continuous functions. Now we shall do the same thing but with integrable functions instead. We let  $\xi_0 = \xi/c_0$  and denote by  $B$  the linear operator in  $L^1([0, \xi_0]; \mathbb{C})$  with domain

$$\mathcal{D}(B) = \{ v \in \mathcal{AC}([0, \xi_0]; \mathbb{C}) \mid v(0) = 0 \}$$

and

$$(Bv)(x) = v'(x), \quad x \in [0, \xi_0], \quad v \in \mathcal{D}(B).$$

As the norm in  $\mathcal{D}(B)$  we can take  $\|w\|_{\mathcal{D}(B)} = \|w'\|_{L^1([0, \xi_0])}$ .

If  $b \in \mathcal{C}(\mathbb{R}^+; \mathbb{R})$  satisfies  $c_0 \leq b(\underline{x}) \leq c_1$ , then we can use an argument similar to the one employed when deriving (35) to conclude that it follows from [1, Thm. 6] that there is a constant  $M_5$  (which depends on  $\alpha, \mu, \tau, \xi, c_0$  and  $c_1$ ) and a unique solution  $v$  of (29) such that

$$\|v_x\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}])}) \leq M_5 \left( \|\chi_{[0, \mathfrak{X}]}(\rho(\underline{y}))h_0(\rho(\underline{y}))\|_{\mathcal{D}_B(\frac{\mu}{\alpha}, \infty)} + \|g\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}])}) \right), \quad (44)$$

for all  $\hat{T} \in [0, \tau]$  and  $\mathfrak{X} \in [0, \xi]$  where  $h_0(\underline{x}) = b(\underline{x})u'_0(\underline{x}) - g(0, \underline{x})$ ,  $\rho$  is the inverse of the function  $x \mapsto y = \int_0^x \frac{1}{b(s)} ds$ , and where  $\mathcal{D}_B(\frac{\mu}{\alpha}) = (L^1([0, \xi_0]; \mathbb{C}), \mathcal{D}(B))_{\frac{\mu}{\alpha}, \infty}$ . In this argument one extends the functions as constants in the  $t$ -direction and as 0 in the  $x$ -direction (but  $u_0$  is extended as a constant) and changes the  $x$ -variable to the new variable  $y = \int_0^x \frac{1}{b(s)} ds$ .

Having (44), our next goal is to estimate the first term on the right hand side. We claim that if  $h$  is an arbitrary function in  $\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi]; \mathbb{R})$ , which is extended as zero to  $(\xi, \infty)$ , then there is a constant  $M_6 \stackrel{\text{def}}{=} 2c_1^\alpha \xi_0 + 4$ , such that

$$\|\chi_{[0, \mathfrak{X}]}(\rho(\underline{y}))h(\rho(\underline{y}))\|_{\mathcal{D}_B(\frac{\mu}{\alpha}, \infty)} \leq M_6 \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])}, \quad \mathfrak{X} \in [0, \xi]. \quad (45)$$

To see this we argue as follows: Let  $w(\underline{y}) = \chi_{[0, \mathfrak{X}]}(\rho(\underline{y}))h(\rho(\underline{y}))$  and extend this function as 0 on  $(-\infty, 0)$  and let  $t \in (0, 1)$  be arbitrary. Now write  $w = w_1 + w_2$  where  $w_1(\underline{y}) = \int_0^\infty \frac{1}{t} e^{-\frac{r}{t}} (w(\underline{y}) - w(\underline{y} - r)) dr$  and where  $w_2(\underline{y}) = \int_0^{\underline{y}} \frac{1}{t} e^{-\frac{r}{t}} w(\underline{y} - r) dr$ . We note that  $w(\underline{y}) = 0$  when  $\underline{y} < 0$  and when  $\underline{y} > \mathfrak{X}_\rho \stackrel{\text{def}}{=} \int_0^{\mathfrak{X}} \frac{1}{b(s)} ds$ . Because  $\rho$  is Lipschitz continuous with constant  $c_1$  we know that  $|w(\underline{y}) - w(\underline{y} - r)| \leq c_1^\alpha r^\alpha \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])}$  when  $0 \leq r \leq \underline{y} \leq \mathfrak{X}_\rho$ . Furthermore,  $|w(\underline{y}) - w(\underline{y} - r)| \leq \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])}$  when  $0 \leq \underline{y} < r$  or  $\mathfrak{X}_\rho < \underline{y} \leq \mathfrak{X}_\rho + r$  (because then either  $w(\underline{y})$  or  $w(\underline{y} - r)$  vanishes) and  $|w(\underline{y}) - w(\underline{y} - r)| = 0$  otherwise. It follows from these inequalities that  $\|w_1\|_{L^1([0, \xi_0])} \leq (t^\alpha c_1^\alpha \xi_0 \Gamma(1 + \frac{\mu}{\alpha}) + 2t) \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])}$ . Furthermore,  $\|w_2\|_{\mathcal{D}(B)} = \|w'_2\|_{L^1([0, \xi_0])} = \frac{1}{t} \|w_1\|_{L^1([0, \xi_0])}$  because  $w'_2(\underline{y}) = \frac{1}{t} w_1(\underline{y})$ . Thus we see that  $t^{-\frac{\mu}{\alpha}} \|w_1\|_{L^1([0, \xi_0])} + t^{1-\frac{\mu}{\alpha}} \|w_2\|_{\mathcal{D}(B)} \leq M_6 \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])}$  and by the definition of the interpolation space  $\mathcal{D}_B(\frac{\mu}{\alpha}, \infty) = (L^1([0, \xi_0]; \mathbb{C}), \mathcal{D}(B))_{\frac{\mu}{\alpha}, \infty}$  (see e.g., [13, Def. 1.2.2]), this is exactly what we need in order to get (45).

Using (45) we see that (44) implies that the function  $v$  that solves (29) satisfies

$$\|v_x\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}])}) \leq M_5 \left( M_6 \|h\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0, \xi])} + \|g\|_{\mathcal{C}^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}])}) \right), \quad (46)$$

for all  $\hat{T} \in [0, \tau]$  and all  $\mathfrak{X} \in [0, \xi]$ .

Let  $c(\underline{t}, \underline{x}) \stackrel{\text{def}}{=} \sigma'(u(\underline{t}, \underline{x}))$ . By (42) we can choose  $T \in (0, \hat{\tau})$  such that

$$\sup_{\substack{t, s \in [0, \hat{\tau}] \\ |t-s| \leq \hat{\tau}-T}} \sup_{x \in [0, \xi]} |c(t, x) - c(s, x)| \leq \frac{1}{2M_5}. \quad (47)$$

Let  $\hat{T}$  be some arbitrary number in  $(T, \hat{\tau})$ .

Now we rewrite (1) in the form

$$(D_t^\alpha(u - u_0))(t, x) + c(T, x)u_x(t, x) = f(t, x) + (c(T, x) - c(t, x))u_x(t, x) \\ \stackrel{\text{def}}{=} g(t, x), \quad t \in [0, \hat{T}], \quad x \in [0, \xi].$$

Note that this equation is of type (29); hence the estimate (46) may be applied to  $u$  with  $b(\underline{x}) = c(T, \underline{x})$  (and  $b$  extended as a constant for  $x > \xi$ ). Also observe that  $c(T, \underline{x})u'_0(\underline{x}) - g(0, \underline{x}) = c(0, \underline{x})u'_0(\underline{x}) - f(0, \underline{x}) = \sigma'(u_0(\underline{x}))u'_0(\underline{x}) - f(0, \underline{x})$ . Thus we see by (46) that

$$\|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \leq M_7 \\ + M_5 \|\chi_{[T, \hat{T}]}(\underline{t})(c(T, \underline{x}) - c(\underline{t}, \underline{x}))u_x(\underline{t}, \underline{x})\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))}, \quad (48)$$

where  $M_7$  is some constant such that

$$M_5 \|f\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} + M_5 M_6 \|\sigma'(u_0(\underline{x}))u'_0(\underline{x}) - f(0, \underline{x})\|_{C^{(\frac{\mu}{\alpha})}([0, \xi])} \\ + M_5 \|\chi_{[0, T]}(\underline{t})(c(T, \underline{x}) - c(\underline{t}, \underline{x}))u_x(\underline{t}, \underline{x})\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \leq M_7,$$

for all  $\mathfrak{X} \in [0, \xi]$  and for all  $\hat{T} \in (T, \hat{\tau})$ . Now a simple calculation shows that

$$\|\chi_{[T, \hat{T}]}(\underline{t})(c(T, \underline{x}) - c(\underline{t}, \underline{x}))u_x(\underline{t}, \underline{x})\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \\ \leq \sup_{t \in [T, \hat{T}]} \sup_{x \in [0, \xi]} |c(T, x) - c(t, x)| \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \\ + \sup_{\substack{t, s \in [0, \hat{T}] \\ t \neq s}} \int_0^{\mathfrak{X}} \frac{|c(t, x) - c(s, x)|}{|t - s|^\mu} |u_x(s, x)| dx.$$

Invoking this inequality together with (47) in (48) we get

$$\|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \leq 2M_7 \\ + 2M_5 \sup_{\substack{t, s \in [0, \hat{T}] \\ t \neq s}} \int_0^{\mathfrak{X}} \frac{|c(t, x) - c(s, x)|}{|t - s|^\mu} |u_x(s, x)| dx \\ \leq 2M_7 + 2M_5 \sup_{s \in [0, \hat{T}]} \int_0^{\mathfrak{X}} \|c\|_{C^{(\mu)}([0, \hat{T}]; C([0, x]))} |u_x(s, x)| dx. \quad (49)$$

Since  $c(\underline{t}, \underline{x}) = \sigma'(\int_0^{\underline{x}} u_x(\underline{t}, r) dr)$  it follows from (23) that

$$\|c\|_{C^{(\mu)}([0, \hat{T}]; C([0, x]))} \leq c_1 + c_2 \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))}, \quad x \in [0, \xi]. \quad (50)$$

From (49) and (50) it follows that for each  $\mathfrak{X} \in [0, \xi]$  there exists a number  $s(\mathfrak{X}) \in [0, \hat{\tau})$  such that

$$\|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, \mathfrak{X}]))} \leq 1 + 2M_7 + 2c_1 M_5 \sup_{s \in [0, \hat{\tau})} \|u_x(s, \underline{x})\|_{L^1([0, \xi])} \\ + 2M_5 c_2 \int_0^{\mathfrak{X}} \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))} |u_x(s(\mathfrak{X}), x)| dx. \quad (51)$$

By (43) there is a finite set of points  $\{t_j\}_{j=1}^n \subset [0, \hat{\tau})$  such that if  $s \in [0, \hat{\tau})$  then there is an index  $j(s) \in \{1, \dots, n\}$  such that

$$\|u_x(s, \underline{x}) - u_x(t_{j(s)}, \underline{x})\|_{L^1([0, \xi])} \leq \frac{1}{4M_5c_2}. \quad (52)$$

Let  $M_8 = \max\{4M_5c_2, 2 + 4M_7 + 4c_1M_5 \sup_{s \in [0, \hat{\tau})} \|u_x(s, \underline{x})\|_{L^1([0, \xi])}\}$ , (by (43)  $M_8 < \infty$ ). Then we conclude from (51) and (52) that we in fact have

$$\begin{aligned} \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))} &\leq M_8 + M_8 \int_0^x \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))} |u_x(t_{j(s(x))}, x)| dx \\ &\leq M_8 + M_8 \int_0^x \|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))} p(x) dx, \end{aligned}$$

where  $p(x) = \max_{1 \leq j \leq n} |u_x(t_j, x)|$  so that we have  $p \in L^1([0, \xi]; \mathbb{R})$ . But now it follows from Gronwall's inequality that

$$\|u_x\|_{C^{(\mu)}([0, \hat{T}]; L^1([0, x]))} \leq M_8 e^{M_8 \int_0^x p(s) ds} \leq M_8 e^{M_8 \|p\|_{L^1([0, \xi])}}.$$

This inequality combined with (50) contradicts (41) and the proof is complete.  $\square$

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ISSN 0784-3143

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