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Teemu Lukkari

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Teemu Lukkari

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *We show that given a positive and finite Radon measure μ, there is a $p(x)$-superharmonic function u which satisfies*

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu$$

in the sense of distributions. Here $\mathcal{A}$ is an elliptic operator with $p(x)$-type nonstandard growth.

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Correspondence

Teemu.Lukkari@tkk.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FI-02015 TKK, Finland
email:math@tkk.fi http://math.tkk.fi/

1 Introduction

We study the existence of solutions of

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu, \tag{1.1}$$

where $\mathcal{A}$ is an operator with $p(x)$-type nonstandard structural conditions. Our main result is that for positive, finite Radon measures μ, there exists a $p(x)$-superharmonic function u which satisfies (1.1) in the sense of distributions. Examples of the operators $\mathcal{A}$ considered here arise from variational integrals like

$$\int |\nabla u|^{p(x)}\,\mathrm{d}x; \tag{1.2}$$

the Euler-Lagrange equation of (1.2) is the $p(x)$-Laplacian equation

$$\operatorname{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1.3}$$

where

$$\mathcal{A}(x,\xi) = p(x)|\xi|^{p(x)-2}\xi.$$

There is an extensive literature on partial differential equations and calculus of variations with various nonstandard growth conditions, see for example [26, 27, 21, 2, 1, 3] and the references in the survey [23].

We study this problem for two reasons. First, some properties of $p(x)$-superharmonic functions require an additional integrability assumption. We would like to show the existence of $p(x)$-superharmonic functions for which the integrability assumption can be verified. The need for an extra assumptions is due to the fact that Harnack estimates are intrinsic in the sense that they depend on the solution itself.

Second, we would like to show the existence of solutions with nonremovable isolated singularities. There is a method due to Serrin [25] to construct such solutions. Again because of the intrinsic nature of the Harnack estimates, this method fails. Hence the second purpose of this work is to find an alternative to Serrin's method. This turns out to be quite simple, just choosing the Dirac measure as μ in (1.1) suffices.

Our approach is an adaptation of that of Kilpeläinen and Malý [15]. First, we obtain approximative solutions u_i by approximating μ with more regular measures. Then we prove uniform estimates for u_i and use them to find a limit u and to prove the fact that the left hand side of (1.1) makes sense as a distribution. Finally we show that this u is indeed a solution of (1.1). This approach is related to the works of Boccardo and Gallouët [4, 5]; see also [6, 20].

The results we use as tools here do not hold without additional assumptions on the function $p(\cdot)$. Even the variable exponent Lebesgue and Sobolev spaces have very few properties for general, for instance just measurable, exponents. There is a frequently used assumption, called logarithmic Hölder continuity, that seems to be the right one for our purposes. See below for more details.

2 Preliminaries

A measurable function $p\colon \mathbb{R}^n \to (1,\infty)$, $n \geq 2$, is called a variable exponent. We denote

$$p_A^+ = \sup_{x\in A} p(x), \quad p_A^- = \inf_{x\in A} p(x), \quad p^+ = \sup_{x\in\mathbb{R}^n} p(x), \quad p^- = \inf_{x\in\mathbb{R}^n} p(x).$$

We will always assume that the exponent $p(\cdot)$ *is logarithmically Hölder continuous, i.e. satisfies* (2.1) *below and that* $1 < p^- \leq p^+ < n$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$$

is finite. The Luxemburg norm on this space is defined as

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\,\mathrm{d}x \leq 1\Big\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. For basic results on variable exponent spaces we refer to [17].

The dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise, [17, Theorem 2.6]. It follows that $L^{p(\cdot)}(\Omega)$ is reflexive. Furthermore, a version of Hölder's inequality,

$$\int_\Omega fg\,\mathrm{d}x \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Smooth functions are not be dense in $W^{1,p(\cdot)}(\Omega)$ without additional assumptions on the exponent $p(\cdot)$. This was observed by Zhikov [26, 27] in the context of the Lavrentiev phenomenon. Zhikov introduced the logarithmic Hölder continuity condition,

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)} \tag{2.1}$$

for all $x, y \in \Omega$ such that $|x-y| \leq 1/2$, to characterise functionals for which the Lavrentiev phenomenon does not occur. If the exponent satisfies (2.1), smooth functions are dense in variable exponent Sobolev spaces and we can define the Sobolev space with zero boundary values, denoted by $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We refer to [9] and [14] for the details of this definition.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Since we assume the exponent $p(\cdot)$ to be continuous, the $p(\cdot)$-*Poincaré inequality*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$, see [11, Theorem 4.1].

We use the following compactness properties of $W^{1,p(\cdot)}(\Omega)$ in our existence proof. The first property follows from the reflexivity of $L^{p(\cdot)}(\Omega)$, the second from the fact that $W^{1,p(\cdot)}(\Omega)$ embeds compactly into $L^{\kappa p(\cdot)}(\Omega)$ for some $\kappa > 1$, [17, Theorem 3.3], and the third from Mazur's lemma.

Theorem 2.2. *Assume that the sequence (u_j) is bounded in $W^{1,p(\cdot)}(\Omega)$. Then we there is a function $v \in W^{1,p(\cdot)}(\Omega)$ and a subsequence (u_{j_k}) with the following properties.*

1. *$\nabla u_{j_k} \to \nabla v$ weakly in $L^{p(\cdot)}(\Omega)$.*
2. *$u_{j_k} \to v$ pointwise almost everywhere and in $L^{p(\cdot)}(\Omega)$.*
3. *If $u_j \in W_0^{1,p(\cdot)}(\Omega)$, $j = 1, 2, \ldots$, then $v \in W_0^{1,p(\cdot)}(\Omega)$.*

We need the following assumptions to hold for the operator $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$.

1. $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$,
2. $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for all $x \in \Omega$,
3. $\mathcal{A}(x,\xi) \cdot \xi \geq \alpha|\xi|^{p(x)}$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
4. $|\mathcal{A}(x,\xi)| \leq \beta|\xi|^{p(x)-1}$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
5. $(\mathcal{A}(x,\eta) - \mathcal{A}(x,\xi)) \cdot (\eta - \xi) > 0$ for all $x \in \Omega$ and $\eta \neq \xi \in \mathbb{R}^n$.

These are called the structure conditions of $\mathcal{A}$.

We say that a function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a subsolution of the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{2.3}$$

if

$$\int_\Omega \mathcal{A}(x,\nabla u) \cdot \nabla\varphi \, \mathrm{d}x \leq 0$$

for all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$. Further, u is a supersolution if $-u$ is a subsolution and a solution if u is both a super- and a subsolution. Since smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$, we are allowed to employ test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ by the usual approximation argument.

Logarithmic Hölder continuity plays an important role in the calculus of variations and theory of partial differential equations with $p(\cdot)$-growth.

Indeed, higher integrability [27], Hölder regularity results [1, 7], and Harnack estimates [2, 12, 10] use condition (2.1). Harnack estimates are used to prove the properties of supersolutions and superharmonic functions we employ here. Hence the log-Hölder assumption is crucial to us.

We say that a function $u : \Omega \to (-\infty, \infty]$ is $p(x)$-superharmonic in Ω, denoted $u \in \mathcal{S}(\Omega)$, if

1. u is lower semicontinuous,
2. u is finite almost everywhere and
3. The comparison principle holds: Let $D \Subset \Omega$ be an open set. If h is a solution in D, continuous in $\overline{D}$ and $u \geq h$ on ∂D, then $u \geq h$ in D.

Further, we say that u is $p(x)$-hyperharmonic if u is either $p(x)$-superharmonic or identically ∞.

If u is a supersolution, then the lower semicontinuous regularization of u, defined as $\widetilde{u}(x) = \operatorname{ess\,liminf}_{y \to x} u(y)$, is a $p(x)$-superharmonic function and equals $u(x)$ for almost every x, see [10, Theorem 17] and [12, Theorem 4.1]. If (u_k) is an increasing sequence of $p(x)$-superharmonic functions, then the limit function is $p(x)$-hyperharmonic. Another consequence of the definition is that if u is $p(x)$-superharmonic, so is the function $\min(u, \lambda)$ for all $\lambda \in \mathbb{R}$. Since bounded $p(x)$-superharmonic functions are supersolutions, [10, Corollary 21], we see that the functions $\min(u, \lambda)$ are supersolutions. We can use this observation about truncations to prove that $p(x)$-superharmonicity is a local property in the same way as in [13, Theorem 7.27].

For a $p(x)$-superharmonic function u we define a derivative Du pointwise as

$$Du = \lim_{k \to \infty} \nabla \min(u, k).$$

Note that Du is not necessarily the gradient of u in any sense.

We recall the following integrability lemma [15, Lemma 1.11]. The proof can also be found in [13].

Lemma 2.4. *Let Ω be bounded, $1 < p < \infty$ and let u be a nonnegative function which is finite almost everywhere. Suppose that for all $k \in \mathbb{N}$*

$$\min(u, k) \in W_0^{1,p}(\Omega)$$

and

$$\int_\Omega |\nabla \min(u, k)|^p \, \mathrm{d}x \leq Mk$$

for a constant M independent of k. If $1 \leq q \leq n/(n-1)$, then

$$\int_\Omega |\nabla \min(u, k)|^{q(p-1)} \, \mathrm{d}x \leq C,$$

where $C = C(n, p, q, M, \operatorname{diam} \Omega)$.

The previous lemma is used to prove the following result; see [10, Theorem 26] and [19, Theorem 4.6]. The extra assumption mentioned in the introduction is the requirement that $u \in L^t_{loc}(\Omega)$.

Theorem 2.5. *Assume that u is $p(x)$-superharmonic in Ω. If $u \in L^t_{loc}(\Omega)$ for some $t > 0$, there is a number $q > 1$ such that $|u|^{q(p(x)-1)}$ and $|Du|^{q(p(x)-1)}$ are locally integrable.*

3 Compactness of $p(x)$-superharmonic functions

In this section we prove a weak compactness property of $p(x)$-superharmonic functions, Theorem 3.4. It is our main tool for the next section.

Lemma 3.1. *Assume that u is a nonnegative subsolution and $\eta \in C_0^\infty(\Omega)$ is such that $0 \le \eta \le 1$. Then*

$$\int_\Omega |\nabla u|^{p(x)} \eta^{p^+} \, \mathrm{d}x \le C \int_\Omega u^{p(x)} |\nabla \eta|^{p(x)} \, \mathrm{d}x.$$

Proof. We use $u\eta^{p^+}$ as a test function and obtain

$$0 \ge \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \, \mathrm{d}x + \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \eta \eta^{p^+ - 1} u \, \mathrm{d}x.$$

From this we obtain that

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \, \mathrm{d}x \le \int_\Omega |\mathcal{A}(x, \nabla u)| |\nabla \eta| \eta^{p^+ - 1} u \, \mathrm{d}x. \tag{3.2}$$

Next we use structure, (3.2) and Young's inequality and conclude that

$$\begin{aligned}
\int_\Omega \eta^{p^+} |\nabla u|^{p(x)} \, \mathrm{d}x \le & C \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \, \mathrm{d}x \\
\le & C \int_\Omega |\mathcal{A}(x, \nabla u)| |\nabla \eta| \eta^{p^+ - 1} u \, \mathrm{d}x \\
\le & C \int_\Omega |\nabla u|^{p(x)-1} |\nabla \eta| \eta^{p^+ - 1} u \, \mathrm{d}x \\
\le & \frac{1}{2} \int_\Omega |\nabla u|^{p(x)} \eta^{\frac{p^+}{p'(x)} p'(x)} \, \mathrm{d}x \\
& + C \int_\Omega u^{p(x)} |\nabla \eta|^{p(x)} \eta^{(p^+ - 1 - \frac{p^+}{p'(x)}) p(x)} \, \mathrm{d}x,
\end{aligned}$$

from which the claim follows. ☐

It follows from the inequalities between the Luxemburg norm and the modular [8, Theorem 1.3] that there is an exponent $s > 0$ such that

$$\| |u|^{p(\cdot)-1} \|_{p'(\cdot)} \le \|u\|^s_{p(\cdot)}. \tag{3.3}$$

We preserve the letter s for this exponent. The exact value of s is not important to us.

Theorem 3.4. *Let (u_j) be a sequence of positive $p(x)$-superharmonic functions. Then there exist a subsequence (u_{j_k}) and a hyperharmonic function u such that $u_{j_k} \to u$ almost everywhere in Ω and $Du_{j_k} \to Du$ almost everywhere in the set $\{u < \infty\}$.*

Proof. Assume first that $u_j \le M < \infty$. Then the functions u_j are supersolutions, [10, Corollary 21]. Let $U \Subset U' \Subset \Omega' \Subset \Omega$ be open sets and choose $\varphi \in C_0^\infty(\Omega')$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in U'. Since $M - u_j$ is a non-negative subsolution, by the Caccioppoli estimate (Lemma 3.1) we obtain

$$\begin{aligned}\int_{U'} |\nabla u_j|^{p(x)}\,\mathrm{d}x \le& \int_{\Omega'} |\nabla u_j|^{p(x)}\varphi^{p^+}\,\mathrm{d}x\\ =& \int_{\Omega'} |\nabla (M-u_j)|^{p(x)}\varphi^{p^+}\,\mathrm{d}x\\ \le& \int_{\Omega'} |u_j - M|^{p(x)}|\nabla\varphi|^{p(x)}\\ \le& CM^{p^+}\int_{\Omega'} |\nabla\varphi|^{p(x)}\,\mathrm{d}x.\end{aligned}$$

We combine this with the $p(\cdot)$-Poincaré inequality and conclude that the sequence (φu_j) is bounded in $W_0^{1,p(\cdot)}(\Omega')$. Thus by Theorem 2.2 there is a function $u \in W_0^{1,p(\cdot)}(\Omega')$ and a subsequence, still denoted by (u_j), such that $u_j \to u$ in $L^{p(\cdot)}(U)$ and pointwise almost everywhere, and finally $\nabla u_j \to \nabla u$ weakly in $L^{p(\cdot)}(U)$.

Next we claim that u has a representative which is $p(x)$-superharmonic in U. To prove this, set $v_i = \inf_{i\le j} u_j$ and for a fixed i, $w_k = \min_{i\le j\le k} u_j$. Then w_k is a supersolution by [10, Theorem 2] and the sequence (w_k) is decreasing and bounded below. By [10, Theorem 12] this implies that $v_i = \lim_{k\to\infty} w_k$ is a supersolution. Thus the function $\widetilde{v}_i(x) = \operatorname{ess\,liminf}_{y\to x} v_i$ is $p(x)$-superharmonic in U'. Let $\widetilde{v} = \lim_{i\to\infty}\widetilde{v}_i$. Now $\widetilde{v}$ is the desired representative since it is $p(x)$-superharmonic as an increasing limit of $p(x)$-superharmonic functions and

$$u(x) = \lim_{j\to\infty} u_j(x) = \lim_{i\to\infty} v_i(x) = \widetilde{v}(x)$$

for almost every $x \in U$.

The next step is to prove that we can assume $\nabla u_j \to \nabla u$ almost everywhere in U by passing to a subsequence. To this end, fix a number $\varepsilon > 0$ and let

$$\begin{aligned}E_j =& \{x \in U : \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla u_j)\cdot(\nabla u - \nabla u_j) \ge \varepsilon\},\\ E_j^1 =& \{x \in E_j : |u - u_j| \ge \varepsilon^2\}\end{aligned}$$

and $E_j^2 = E_j \setminus E_j^1$. $|E_j^1| \to 0$ as $j \to \infty$ since $u_j \to u$ in $L^{p(\cdot)}(U)$. To estimate $|E_j^2|$, we note that

$$|E_j^2| \le \frac{1}{\varepsilon}\int_{E_j^2} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla u_j))\cdot(\nabla u - \nabla u_j)\,\mathrm{d}x,$$

pick a function $\eta \in C_0^\infty(U')$ such that $0 \le \eta \le 1$, $\eta = 1$ in U and set

$$v_j = \min((u_j - u + \varepsilon^2)^+, 2\varepsilon^2).$$

We use ηv_j as a test function and obtain

$$0 \le \int_{U'} \mathcal{A}(x, \nabla u) \cdot v_j \nabla \eta \, \mathrm{d}x + \int_{U' \cap \{|u-u_j|<\varepsilon^2\}} \mathcal{A}(x, \nabla u) \cdot \eta(\nabla u_j - \nabla u) \, \mathrm{d}x. \quad (3.5)$$

Since $\nabla u_j \to \nabla u$ weakly, we conclude that

$$\|\nabla u\|_{p(\cdot),U'} \le \liminf_{j\to\infty} \|\nabla u_j\|_{p(\cdot),U'} \le C \quad (3.6)$$

by the Caccioppoli estimate. We use (3.5), structure of $\mathcal{A}$, the Hölder inequality, (3.3) and (3.6) and get that

$$\begin{aligned}
\int_{U' \cap \{|u-u_j|<\varepsilon^2\}} \mathcal{A}(x, \nabla u) \cdot \eta(\nabla u - \nabla u_j) \, \mathrm{d}x \le& \int_{U'} \mathcal{A}(x, \nabla u) \cdot v_j \nabla \eta \, \mathrm{d}x \\
\le& C\varepsilon^2 \int_{U'} |\nabla u|^{p(x)-1} |\nabla \eta| \, \mathrm{d}x \\
\le& C\varepsilon^2 \|\nabla u\|_{p(\cdot),U'}^s \|\nabla \eta\|_{p(\cdot),U'} \\
\le& C\varepsilon^2.
\end{aligned}$$

Replacing v_j with $\widetilde{v}_j = \min((u_j - u + \varepsilon^2)^+, 2\varepsilon^2)$ allows us to reverse the roles of u_j and u in the above computation. Thus we conclude that

$$|E_j^2| \le \frac{1}{\varepsilon} \int_{E_j^2} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \, \mathrm{d}x \le C\varepsilon.$$

It follows that

$$|E_j| = |E_j^1| + |E_j^2| \le (C+1)\varepsilon \quad (3.7)$$

for $j \ge j_\varepsilon$.

Estimate (3.7) implies that $\nabla u_j \to \nabla u$ in measure; this allows us to pick the desired pointwise almost everywhere convergent subsequence. To prove the convergence in measure, we assume the opposite and find positive numbers δ and a such that

$$|\{x \in U : |\nabla u_j - \nabla u| \ge \delta| \ge a > 0.$$

Pick any sequence (ε_k) such that $\varepsilon_k \to 0$ as $k \to \infty$. We note that

$$\begin{aligned}
&|\{x \in U : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon_k\}| \\
&\qquad \ge |\{x \in U : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon_k, \\
&\qquad\qquad |\nabla u_j - \nabla u| \ge \delta\}|.
\end{aligned}$$

By measure theory, the structure of $\mathcal{A}$ and the counterassumption, the right hand side tends to a limit $L \ge a$ as $k \to \infty$. Thus there is a number $\varepsilon_0 > 0$ such that

$$|\{x \in U : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon\}| \ge a/2 > 0$$

whenever $\varepsilon \leq \varepsilon_0$, and this contradicts (3.7).

We have proved that if the original sequence (u_j) is bounded and $U \Subset \Omega$ we can find a subsequence that converges to a function u which is $p(x)$-superharmonic in U. To find a limit which is $p(x)$-superharmonic in Ω, choose open sets U_k, $k = 1, 2, \ldots$, such that $U_k \Subset U_{k+1}$ and $\Omega = \cup_k U_k$. Then we can pick a subsequence (u_j^1) and a limit function u^1 which is $p(x)$-superharmonic in U_1. We proceed inductively and pick a subsequence (u_j^{k+1}) of (u_j^k) that converges to a function $u^{k+1} \in \mathcal{S}(U_{k+1})$. Then $u_k = u_{k+1}$ almost everywhere in U_k, and by $p(x)$-superharmonicity this holds everywhere. Thus we can define the desired limit function as $u = u_k$ in U_k. u is $p(x)$-superharmonic in Ω since being $p(x)$-superharmonic is a local property. Finally, we note that by construction for the diagonal sequence (u_k^k) it holds that $u_k^k \to u$ and $\nabla u_k^k \to \nabla u$ almost everywhere in Ω.

As the final step we remove the boundedness assumption by another diagonalization argument. By the first part of the theorem, we can find a subsequence (u_j^1) and a $p(x)$-superharmonic function u_1 such that

$$\min(u_j^1, 1) \to u_1 \text{ and } \nabla \min(u_j^1, 1) \to \nabla u_1$$

almost everywhere in Ω. Again we proceed inductively and pick a subsequence (u_j^k) of (u_j^{k-1}) such that

$$\min(u_j^k, k) \to u_k \text{ and } \nabla \min(u_j^k, k) \to \nabla u_k$$

almost everywhere in Ω. We observe that if $l \geq k$ and $u_k(x) < k$, we have $u_l(x) = u_k(x)$. Thus the sequence (u_k) is increasing, and we conclude that the limit $u = \lim_{k\to\infty} u_k$ exists and defines the desired hyperharmonic function in Ω. We note that by construction $\min(u, k) = u_k$, so that for the diagonal sequence (u_k^k) it holds that $\nabla u_k^k \to Du$ almost everywhere in the set $\{u < \infty\}$. □

4 Existence of $p(x)$-superharmonic solutions

In this section we prove our main existence result, Theorem 4.7. Throughout, we use T to denote the map defined by

$$(Tu, \varphi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \,\mathrm{d}x, \tag{4.1}$$

where $\varphi \in C_0^\infty(\Omega)$. By Theorem 2.5 and the structure of $\mathcal{A}$, Tu defines a distribution for integrable $p(x)$-superharmonic functions u and $Tu \in (W^{1,p(\cdot)}(\Omega))^*$ if $u \in W^{1,p(\cdot)}(\Omega)$.

Theorem 4.2. *Let u be an integrable $p(x)$-superharmonic function. Then there is a positive Radon measure μ such that*

$$-\operatorname{div} \mathcal{A}(x, Du) = \mu$$

in the sense of distributions.

Proof. Since u is integrable, $|Du|^{p(x)-1} \in L^1_{loc}(\Omega)$ by Theorem 2.5. Pick any $\varphi \in C_0^\infty(\Omega)$ and denote $u_k = \min(u, k)$. Then

$$\mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi \to \mathcal{A}(x, Du) \cdot \nabla\varphi$$

pointwise almost everywhere by the continuity of $\xi \mapsto \mathcal{A}(x, \xi)$.

Using the structure of $\mathcal{A}$, we have

$$|\mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi| \leq C|\nabla u_k|^{p(x)-1}|\nabla\varphi| \leq C|Du|^{p(x)-1}|\nabla\varphi|.$$

Using the dominated convergence theorem and the fact that the functions u_k are supersolutions, we conclude that

$$\begin{aligned}(Tu, \varphi) &= \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \, \mathrm{d}x \\ &= \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi \, \mathrm{d}x \geq 0.\end{aligned}$$

The claim now follows from the Riesz representation theorem, see for example [24, Theorem 2.14]. □

Lemma 4.3. *Let $u, v \in W_0^{1,p(\cdot)}(\Omega)$ be supersolutions such that*

$$Tu = \mu \leq \nu = Tv.$$

Then $u \leq v$ almost everywhere in Ω.

Proof. Let $\eta = \min(v - u, 0)$. Since $\mu \leq \nu$, we obtain that

$$\begin{aligned}0 &\geq \int_\Omega \eta \, \mathrm{d}\nu - \int_\Omega \eta \, \mathrm{d}\mu \\ &= \int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla\eta \, \mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\eta \, \mathrm{d}x \\ &= \int_{\{u>v\}} (\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)) \cdot (\nabla v - \nabla u) \, \mathrm{d}x.\end{aligned}$$

By the monotonicity of $\mathcal{A}$, it follows that $\nabla v = \nabla u$ almost everywhere in $\{u > v\}$. Hence $\nabla\eta = 0$ and it follows that $\eta = 0$ almost everywhere, which means that $v \geq u$ almost everywhere. □

To show the existence of solutions in the case $\mu \in (W^{1,p(\cdot)}(\Omega))^*$, we use the following theorem. See [18, Théorème 2.1, p. 171] for the proof.

Theorem 4.4. *Let X be a reflexive, separable Banach space, and assume that $T : X \to X^*$ is*

1. *monotone, i.e. $\langle Tu - Tv, u - v\rangle \geq 0$ for all $u, v \in X$,*
2. *bounded, i.e. if $E \subset X$ is bounded, so is $T(E)$;*

3. demicountinuous, i.e. $x_j \to x$ implies $(Tx_j, y) \to (Tx, y)$ for all $y \in X$ and

4. coercive, i.e. for a sequence $(x_j) \subset X$ such that $\|x_j\|_X \to \infty$ it holds that

$$\frac{(Tx_j, x_j)}{\|x_j\|_X} \to \infty \text{ as } j \to \infty.$$

Then T is surjective, i.e. the equation $Tx = f$ has a solution $x \in X$ for each $f \in X^*$.

Theorem 4.5. *Let Ω be a bounded domain and $\mu \in (W_0^{1,p(\cdot)}(\Omega))^*$ be a positive Radon measure. Then there is a unique nonnegative supersolution $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in the sense of distributions.

Proof. We prove the existence part by verifying the assumptions of Theorem 4.4 for the map $T : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ given by (4.1). First, the monotonicity of T is an immediate consequence of the monotonicity assumption on $\mathcal{A}$.

Using the structure of $\mathcal{A}$, the Hölder inequality and (3.3), we infer that

$$\begin{aligned} |(Tu, v)| \leq & C \int_\Omega |\nabla u|^{p(x)-1} |\nabla v| \, \mathrm{d}x \\ \leq & C \| |\nabla u|^{p(x)-1} \|_{p'(\cdot)} \|\nabla v\|_{p(\cdot)} \\ \leq & C \|u\|_{1,p(\cdot)}^s \|v\|_{1,p(\cdot)}. \end{aligned}$$

This implies that $\|Tu\|_{(W_0^{1,p(\cdot)}(\Omega))^*} \leq C\|u\|_{1,p(\cdot)}$, so that T is bounded.

Let $(u_j) \subset W_0^{1,p(\cdot)}(\Omega)$ be such that $u_j \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. We pass to a subsequence and assume that $u_j \to u$ and $\nabla u_j \to \nabla u$ pointwise almost everywhere. By continuity of the map $\xi \mapsto \mathcal{A}(x, \xi)$, it follows that $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, \nabla u)$ almost everywhere. Since

$$\int_\Omega |\mathcal{A}(x, \nabla u_j)|^{p(x)/(p(x)-1)} \, \mathrm{d}x \leq C \int_\Omega |\nabla u_j|^{p(x)} \, \mathrm{d}x \leq M < \infty$$

by the convergence of the sequence (u_j), $(\mathcal{A}(x, \nabla u_j))$ is bounded in $L^{p'(\cdot)}(\Omega)$. Thus we may pass to a further subsequence and assume that $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, \nabla u)$ weakly in $L^{p'(\cdot)}(\Omega)$.

This implies that the whole sequence converges weakly; indeed, assuming the opposite, we find a weak neighbourhood U of $\mathcal{A}(x, \nabla u)$ and a subsequence such that $(\mathcal{A}(x, \nabla u_{j_k})) \subset L^{p'(\cdot)}(\Omega) \setminus U$. We may assume pointwise convergence by passing to a further subsequence, and this sub-subsequence converges weakly in $L^{p'(\cdot)}(\Omega)$ to $\mathcal{A}(x, \nabla u)$ by the earlier argument, which is a contradiction. It follows that

$$(Tu_j, v) = \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla v \, \mathrm{d}x \to \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v \, \mathrm{d}x = (Tu, v).$$

Let (u_j) be a sequence such that $\|u_j\|_{1,p(\cdot)} \to \infty$. We may assume that $\|u_j\|_{1,p(\cdot)} \geq 1$, so that $\varrho_{p(\cdot)}(\nabla u_j) \geq \|\nabla u_j\|_{p(\cdot)}^{p^-}$, by [8, Theorem 1.2]. We use the structure of $\mathcal{A}$, and the $p(\cdot)$-Poincaré inequality and obtain that

$$\frac{(Tu_j, u_j)}{\|u_j\|_{1,p(\cdot)}} \geq C \frac{\int_\Omega |\nabla u_j|^{p(x)} \, \mathrm{d}x}{\|u_j\|_{1,p(\cdot)}} \geq C \frac{\|\nabla u_j\|_{p(\cdot)}^{p^-}}{\|u_j\|_{1,p(\cdot)}} \geq C \|u_j\|_{1,p(\cdot)}^{p^- - 1} \to \infty$$

as $j \to \infty$.

Finally, we note that the uniqueness and positivity claims follow from Lemma 4.3. □

We say that a sequence of measures (μ_j) converges weakly to a measure μ if

$$\lim_{j \to \infty} \int_\Omega \varphi \, \mathrm{d}\mu_j = \int_\Omega \varphi \, \mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega)$. Approximation using convolution is tricky in our setting, since the spaces $L^{p(\cdot)}(\Omega)$ are usually not translation invariant. We use the following technique from Mikkonen's thesis [22] to overcome this difficulty.

Lemma 4.6. *Let Ω be a bounded open set and assume that μ is a finite Radon measure on Ω. Then there is a sequence (μ_j) of finite Radon measures such that $\mu_j \in (W_0^{1,p(\cdot)}(\Omega))^*$, $\mu_j \to \mu$ weakly and $\mu_j(\Omega) \leq \mu(\Omega)$.*

Proof. Let $Q_{i,j}$, $i = 1, \ldots, N_j$, be the dyadic cubes with side length 2^{-j} contained in Ω. For $E \subset \Omega$ we define

$$\mu_j(E) = \sum_{i=1}^{N_j} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} |E \cap Q_{i,j}|,$$

and the proof will be completed by showing that the sequence (μ_j) has the desired properties.

First we observe that

$$\mu_j(\Omega) = \sum_{i=1}^{N_j} \mu(Q_{i,j}) \leq \mu(\Omega),$$

since the cubes $Q_{i,j}$ do not completely cover the set Ω.

Given a function $\varphi \in C_0^\infty(\Omega)$ we obtain

$$\begin{aligned} \left| \int_\Omega \varphi \, \mathrm{d}\mu_j \right| &= \left| \sum_{i=1}^{N_j} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} \int_{Q_{i,j}} \varphi \, \mathrm{d}x \right| \\ &\leq \frac{\mu(\Omega)}{\min |Q_{i,j}|} \int_\Omega |\varphi| \, \mathrm{d}x \\ &\leq C \frac{\mu(\Omega)}{|\Omega|} \|\varphi\|_{p(\cdot)} \|1\|_{p'(\cdot)} \\ &\leq C \|\varphi\|_{1,p(\cdot)}, \end{aligned}$$

so that $\mu_j \in (W_0^{1,p(\cdot)}(\Omega))^*$.

To establish the weak convergence, let $\varepsilon > 0$ and pick any function $\varphi \in C_0^\infty(\Omega)$. For sufficiently large j, $\operatorname{spt}\varphi \subset \cup_{i=1}^{N_j} Q_{i,j}$ and $|\varphi(x_{i,j}) - \varphi(x)| \le \varepsilon$ for $x_{i,j}, x \in Q_{i,j}$. Now we have

$$\begin{aligned}\int_\Omega \varphi \,\mathrm{d}\mu - \int_\Omega \varphi \,\mathrm{d}\mu_j =& \sum_{i=1}^{N_j} \Big(\int_{Q_{i,j}} \varphi \,\mathrm{d}\mu - \frac{\mu(Q_{i,j})}{|Q_{i,j}|} \int_{Q_{i,j}} \varphi \,\mathrm{d}x \Big) \\ \le& \sum_{i=1}^{N_j} [(\varphi(x_{i,j}) + \varepsilon)\mu(Q_{i,j})) - (\varphi(x_{i,j}) - \varepsilon)\mu(Q_{i,j})] \\ \le& 2\mu(\Omega)\varepsilon.\end{aligned}$$

By a similar computation, we obtain that

$$\Big| \int_\Omega \varphi \,\mathrm{d}\mu - \int_\Omega \varphi \,\mathrm{d}\mu_j \Big| \le 2\mu(\Omega)\varepsilon,$$

and conclude that $\mu_j \to \mu$ weakly. □

Theorem 4.7. *Let Ω be bounded and μ a finite Radon measure. Then there is an integrable $p(x)$-superharmonic function u such that $\min(u,k) \in W_0^{1,p(\cdot)}(\Omega)$ for all $k > 0$ and*

$$-\operatorname{div} \mathcal{A}(x, Du) = \mu$$

in the sense of distributions.

Proof. Let (μ_j) be the sequence of measures belonging to $(W_0^{1,p(\cdot)}(\Omega))^*$ obtained from Lemma 4.6 and denote by (u_j) the sequence of supersolutions satisfying

$$-\operatorname{div} \mathcal{A}(x, \nabla u_j) = \mu_j \tag{4.8}$$

in the sense of distributions; such u_j exist by Theorem 4.5.

By Theorem 3.4, there is a hyperharmonic function u such that we can assume $u_j \to u$ and $\nabla \min(u_j,k) \to \nabla \min(u,k)$ almost everywhere by passing to a subsequence. As the first step, we prove that u is integrable. To this end, using structure of $\mathcal{A}$ and (4.8), we infer that

$$\begin{aligned}\int_\Omega |\nabla \min(u_j,k)|^{p(x)} \,\mathrm{d}x \le& C \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla \min(u_j,k) \,\mathrm{d}x \\ =& C \int_\Omega \min(u_j,k) \,\mathrm{d}\mu_j \\ \le& C\mu_j(\Omega)k \le C\mu(\Omega)k.\end{aligned} \tag{4.9}$$

From (4.9) and the p^--Poincaré inequality, we obtain that

$$\begin{aligned}\int_\Omega |\min(u_j,k)|^{p^-} \,\mathrm{d}x \le& C \int_\Omega |\nabla \min(u_j,k)|^{p^-} \,\mathrm{d}x \\ \le& \int_\Omega |1 + \nabla \min(u_j,k)|^{p(x)} \,\mathrm{d}x \\ \le& C|\Omega| + C\mu(\Omega)k \le C(|\Omega| + \mu(\Omega))k.\end{aligned} \tag{4.10}$$

Since $u_j \to u$ almost everywhere, it follows from Fatou's lemma and (4.10) that

$$\int_\Omega |\min(u,k)|^{p^-}\,\mathrm{d}x \le Mk,$$

with the constant M independent of k. This estimate implies that u is finite almost everywhere. Indeed, denoting $E = \{x \in \Omega : u(x) = \infty\}$, we get

$$|E| = \frac{1}{k^{p^-}} \int_E k^{p^-}\,\mathrm{d}x \le \frac{1}{k^{p^-}} \int_\Omega |\min(u,k)|^{p^-}\,\mathrm{d}x \le Mk^{1-p^-} \to 0$$

as $k \to \infty$. Estimate (4.9) and the $p(\cdot)$-Poicaré inequality imply that $(\min(u_j,k))$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. It follows that $\min(u,k) \in W_0^{1,p(\cdot)}(\Omega)$ since weak limits must coincide with pointwise limits. We use (4.10), pointwise convergence of the gradients and Fatou's lemma and obtain the estimate

$$\int_\Omega |\nabla \min(u,k)|^{p^-}\,\mathrm{d}x \le Mk,$$

and the integrability of u follows, see Lemma 2.4 and Theorem 2.5.

By Theorem 4.2, there is a measure ν such that

$$-\operatorname{div} \mathcal{A}(x, Du) = \nu \tag{4.11}$$

in the sense of distributions. We will complete the proof by showing that $\mu = \nu$. Since weak limits of measures are unique, this follows by proving that $\mu_j \to \nu$ weakly. If $q > 1$ is an exponent allowed in Theorem 2.5, using the structure of $\mathcal{A}$ we get that

$$\int_\Omega |\mathcal{A}(x, \nabla u_j)|^q\,\mathrm{d}x \le C \int_\Omega |\nabla u_j|^{q(p(x)-1)}\,\mathrm{d}x \le C.$$

Thus $\mathcal{A}(\cdot, \nabla u_j) \to \mathcal{A}(\cdot, Du)$ weakly in $L^q_{loc}(\Omega)$. We use (4.8), the weak convergence in L^q and (4.11) and conclude that

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega \varphi\,\mathrm{d}\mu_j &= \lim_{j\to\infty} \int_\Omega \mathcal{A}(x, \nabla u_j)\cdot\nabla\varphi\,\mathrm{d}x \\ &= \int_\Omega \mathcal{A}(x, Du)\cdot\nabla\varphi\,\mathrm{d}x \\ &= \int_\Omega \varphi\,\mathrm{d}\nu,\end{aligned}$$

and the proof is complete. □

5 Solutions with isolated singularities

In this section we show the existence of solutions with nonremovable isolated singularities. We assume without loss of generality that the origin belongs to Ω and use δ to denote the unit mass at the origin.

Theorem 5.1. *If u is a solution of*

$$-\operatorname{div} \mathcal{A}(x, Du) = \delta \tag{5.2}$$

obtained from Theorem 4.7, then u is a solution of

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{5.3}$$

in $\Omega \setminus \{0\}$.

Proof. Let (μ_j) be the sequence approximating δ we obtain from Lemma 4.6. From the proof the lemma we see that the support of μ_j is contained in a ball $B_j = B(0, c\,2^{-j})$, where the constant c is independent of j. Thus the corresponding supersolution u_j is a solution of (5.3) in $\Omega \setminus \overline{B}_j$. Note also that the subsequence of Theorem 3.4 is increasing. Thus for each ball $B \Subset \Omega \setminus \{0\}$ we can find an increasing sequence (u_k) of solutions such that $u = \lim_{k\to\infty} u_k$. It follows from the integrability of u and Harnack's principle [10, Theorem 16] that u is a solution in B, and hence also in $\Omega \setminus \{0\}$ since being a solution is a local property. □

The above proof can be easily modified to show that a solution of

$$-\operatorname{div} \mathcal{A}(x, Du) = \mu$$

constructed by the present method is a solution of (5.3) in $\Omega \setminus \operatorname{spt}(\mu)$. However, solutions of equations involving measures are not necessarily unique without some additional assumptions, even when the exponent is constant; see [16] for an example. Hence our present tools are insufficient to obtain the conclusion of Theorem 5.1 for an arbitrary solution of (5.2).

A solution of (5.2) cannot be a supersolution of (5.3). This follows from the growth estimate

$$u(x) \geq C r^{-(n-p^-_{B_R})/(p^+_{B_R}-1)} \tag{5.4}$$

where $|x| = r < 2R$ [19, Corollary 4.15]. Indeed, denoting $q = (n - p^-_{B_R})/(p^+_{B_R} - 1)$ and using (5.4) we have

$$|\nabla u| \geq |\frac{\partial u}{\partial r}| \geq C r^{q-1}.$$

A computation shows that $|\nabla u|$ is *not* integrable to the power $p^-_{B_R}$, if $p^-_{B_R}/n < 1 - \operatorname{osc}_{B_R} p$. The left hand side tends to a limit smaller than one while the right hand tends to one as $R \to 0$ since $p(\cdot)$ is continuous. Thus by choosing a sufficiently small R we see that $|\nabla u|$ does not belong to $L^{p(\cdot)}_{loc}(\Omega)$.

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