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Abstract: *In this paper we present a sufficient condition for the validity of a discrete maximum principle (DMP) for a class of elliptic problems of the second order with a nondiagonal coefficient matrix, solved by means of linear finite elements (FEs). Numerical tests are presented.*

AMS subject classifications: 65N30, 65N50

Keywords: discrete maximum principle, finite element method, acute triangulations, full diffusion tensor

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1 Introduction

There are many works devoted to the validity of various DMPs for FE-type approximations. In general, all such papers can be split into two groups: those dealing with standard (linear) FE computational schemes, see e.g. [2], [5], [8], [9], [21], and papers, where certain nonlinear FE-type schemes are proposed, as in [4], [11], [15], [16].

There are two main reasons why nonlinear FE schemes are developed. First of all, linear FE schemes have been shown to produce approximations satisfying DMP for problems mostly with diagonal coefficient matrices. Second, such schemes often require a usage of FE triangulations with certain geometrical properties (e.g. the nonobtuseness of simplicial FE meshes, etc.). However, it is not always easy to construct such meshes, and further refine them preserving the desired geometrical conditions.

Note that the only obtuse triangle in a given triangulation may destroy the validity of DMP when solving the Poisson equation by standard linear finite elements, see [3]. The situation is even worse for anisotropic case.

In this paper we demonstrate that imposing slightly more severe condition on the triangulations used (forcing them be acute and not only nonobtuse as in [2], [5], [8], [9]), we can still produce DMP-adequate approximations for some class of elliptic problems with full diffusion tensors using only standard linear FE schemes, see [7] for treating similar situation in the parabolic case. We need to solve elliptic equations with nondiagonal coefficient matrices in many areas, e.g., for flow in porous media, transport of atmospheric gases, heat conduction in anisotropic media, financial mathematics [6], [13], [18], [19].

2 Model problem and maximum principle

We consider the following elliptic problem: Find a function u such that

$$-\mathrm{div}\,(\mathcal{A}\nabla u) = f \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded polygonal domain with Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$, $\mathcal{A}$ is a symmetric uniformly positive definite 2×2 matrix (often called a diffusion tensor) with smooth entries $\mathcal{A}_{km},\ k, m = 1, 2$, defined on Ω.

The classical solution of (1)–(2), if it exists, is known to satisfy the following maximum principle [12]:

$$f \le 0 \qquad \Longrightarrow \qquad u \le 0 \quad \text{in} \quad \overline{\Omega}. \tag{3}$$

Remark 2.1 *If the matrix $\mathcal{A}$ is constant then by the linear transformation $F(x) = \mathcal{A}^{-1/2}x$ equation* (1) *becomes the Poisson equation on the domain $F(\Omega)$ with zero boundary conditions on $\partial(F(\Omega))$.*

3 FE discretization

We shall use the standard Sobolev space notation. Assume that the coefficients $\mathcal{A}_{km} \in L^\infty(\Omega)$. Then the weak formulation of problem (1)–(2) reads: Find a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega). \tag{4}$$

Let $\mathcal{T}_h$ stand for a face-to-face triangulation of $\overline{\Omega}$ with a discretization parameter h and triangular elements denoted by symbol T (possibly with some subindices). Let $V_h \subset H_0^1(\Omega)$ be a FE space spanned by the standard piecewise linear continuous basis functions $\varphi_1, \ldots, \varphi_N$ associated with the interior nodes $P_1, \ldots, P_N$, i.e., those vertices of triangles from $\mathcal{T}_h$ that do not belong to $\partial\Omega$.

The FE solution of problem (4) is a function $u_h \in V_h$ such that

$$\int_\Omega \mathcal{A}\nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \forall v_h \in V_h. \tag{5}$$

Algorithmically, $u_h = \sum_{i=1}^{N} y_i \varphi_i$ with $\mathbf{y} = (y_1, \ldots, y_N)^\top$ being a vector-solution of the following system of linear equations

$$\mathbf{A}\mathbf{y} = \mathbf{F}, \tag{6}$$

where $\mathbf{A}$ is the finite element $N \times N$ matrix with entries $a_{ij} = \int_\Omega \mathcal{A}\nabla\varphi_j \cdot \nabla\varphi_i \, dx$, and the vector $\mathbf{F} = (f_1, \ldots, f_N)^\top$ has entries $f_i = \int_\Omega f \varphi_i \, dx$.

Consider acute triangulations, i.e., for any angle α of any element T from $\mathcal{T}_h$ we have

$$\alpha \leq \frac{\pi}{2} - \alpha_1, \tag{7}$$

where α_1 is a fixed positive constant. Obviously, this implies that

$$\alpha \geq 2\alpha_1 \tag{8}$$

for any angle α of any $T \in \mathcal{T}_h$.

Remark 3.1 *For a given $\mathcal{T}_h$ there exists a constant $C > 0$ such that for any triangle $T \in \mathcal{T}_h$ and any of its altitude ℓ (which is always inside of T due to the acuteness property) we have*

$$\ell \geq C h_T, \tag{9}$$

where h_T is the diameter of T.

4 DMP and conditions for its validity

The following implication

$$f \leq 0 \qquad \Longrightarrow \qquad u_h \leq 0 \quad \text{in} \quad \overline{\Omega} \tag{10}$$

presents a natural discrete analogue (discrete maximum principle, or DMP in short) of the maximum principle (3). It is clear that (10) holds if $\mathbf{A}$ is a monotone matrix.

It is known that $\mathbf{A}$ is symmetric and positive definite. Hence, due to a well-known result on monotonicity of the Stieltjes matrices (cf. [20, p. 85]) we only need to provide the nonpositivity of the off-diagonal entries of $\mathbf{A}$ which, in turn, holds if

$$a_{ij}|_T = \int_T \mathcal{A}\nabla\varphi_j \cdot \nabla\varphi_i \, dx \leq 0 \quad (i \neq j) \tag{11}$$

for each triangular element $T \in \mathcal{T}_h$.

In what follows, we shall work with the following matrix decompositions of the diffusion tensor

$$\mathcal{A} = \mathcal{D} + \mathcal{B}, \tag{12}$$

where $\mathcal{D} = d\,I$, with I being the unit matrix and $d > 0$ being a positive constant. The entries of $\mathcal{B} = \mathcal{A} - \mathcal{D}$ are denoted by $\mathcal{B}_{km}$, $k, m = 1, 2$. Let

$$\bar{b} := \max_{k,m=1,2} \{\operatorname{ess\,sup}_{x\in\Omega} |\mathcal{B}_{km}(x)|\}. \tag{13}$$

Remark 4.1 *Note that there are infinitely many decompositions of the type (12).*

Theorem 4.1 *Let*

$$\bar{b} \leq \frac{d\,C^2 \operatorname{ctg} 2\alpha_1}{4\cos\alpha_1}, \tag{14}$$

where the constants α_1, C, and $\bar{b}$ are defined in (7), (9), and (13), respectively. Then the discrete maximum principle (10) holds.

P r o o f: We observe that, in view of (12),

$$a_{ij}|_T = \int_T \mathcal{A}\nabla\varphi_j \cdot \nabla\varphi_i \, dx = \int_T (\mathcal{D}+\mathcal{B})\nabla\varphi_j \cdot \nabla\varphi_i \, dx = d\int_T \nabla\varphi_j \cdot \nabla\varphi_i \, dx +$$

$$+ \int_T \mathcal{B}\nabla\varphi_j \cdot \nabla\varphi_i \, dx \leq d\int_T \nabla\varphi_j \cdot \nabla\varphi_i \, dx + 4\bar{b}\int_T \Big(\max_{i=1,\dots,N,k=1,2}\Big|\frac{\partial\varphi_i}{\partial x_k}\Big|\Big)^2 dx. \tag{15}$$

The following well-known formula

$$\int_T \nabla\varphi_i \cdot \nabla\varphi_j \, dx = -\frac{1}{2}\operatorname{ctg}\alpha_{ij}^T \tag{16}$$

is valid, where α_{ij}^T is the angle in T opposite to the edge P_iP_j.

Further, from (9) we have the following property for the basis functions

$$\left|\frac{\partial\varphi_i}{\partial x_k}\Big|_T\right| \le \frac{1}{Ch_T}, \tag{17}$$

where C is the constant from (9), $i = 1,\dots,N$, and $k = 1,2$. Therefore, using (15), (16), (17), (7), and the formula for the area of a triangle, we get

$$a_{ij}|_T \le -\frac{d}{2}\operatorname{ctg}\alpha_{ij}^T + \frac{2\bar{b}}{C^2}\sin(\frac{\pi}{2}-\alpha_1) \le -\frac{d}{2}\operatorname{ctg}2\alpha_1 + \frac{2\bar{b}}{C^2}\cos\alpha_1 \le 0,$$

provided (14) holds. ■

Remark 4.2 *Consider a strongly regular family of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$, see [1] for a definition. The same analysis can be now done again with another (uniform) constants.*

5 Numerical tests

Consider our problem with the diffusion tensor $\mathcal{A}$ defined by

$$\mathcal{A}(x) = \begin{bmatrix} r_2(x) & r_1(x) \\ -r_1(x) & r_2(x) \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & 1/K \end{bmatrix} \begin{bmatrix} r_2(x) & -r_1(x) \\ r_1(x) & r_2(x) \end{bmatrix}, \tag{18}$$

where $(r_1(x), r_2(x))^\top$ is the unit (normalized) radius-vector from the origin to the point x. The condition number of $\mathcal{A}(x)$ and the eccentricity of the associated ellipse is determined by a positive parameter K. The diffusion tensor field is radially symmetric, so it should behave in the same manner in every direction. Such a problem may describe, e.g., the temperature distribution in a wooden log or in a vulcanic basalt.

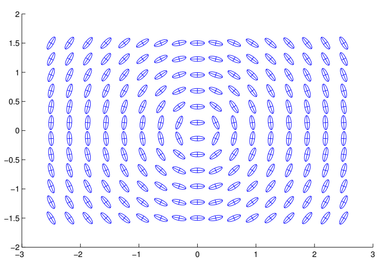

Figure 1: Diffusion tensor field. The main axes of ellipsae indicate directions of the largest heat conductivities (for $K = 3$).

The shape of the domain $\Omega(s)$ is controlled by a slope parametr s. The rectangular domain $(-2.5, 2.5)\times(-1.5, 1.5)$ is discretized in the standard way (see Fig. 2) and then it is deformed so that each subsequent row of elements

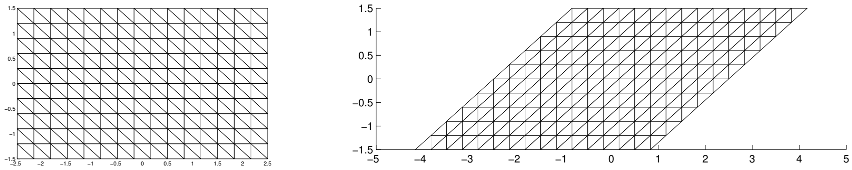

Figure 2: The triangulation for $s = 0$ and $s = 1$.

is shifted against the preceding one by the distance sh_x, where h_x is width of an element in direction of the axis x. Thus,

$$\Omega(s) = \{(x, y) \in R^2 : y \in (-1.5, 1.5);\ -\frac{5}{2} + s\frac{10}{9}y \leq x \leq \frac{5}{2} + s\frac{10}{9}y\}.$$

If $s \in (0, 1)$, the triangulation has only acute angles. For $s = 0.5$ all elements are isosceles triangles. For $s = 0$ and $s = 1$ they have right angles and the triangulations have locally the same structure, but the shape of the domain is different.

We choose the parameter d from decomposition (12) as the mean value of diagonal entries of the middle matrix on the right-hand side of (18), i.e., $d = \frac{K^2+1}{2K}$.

If the DMP is not satisfied, the inverse of the stiffness matrix has to contain some negative entries (cf. Fig. 3 (left)).

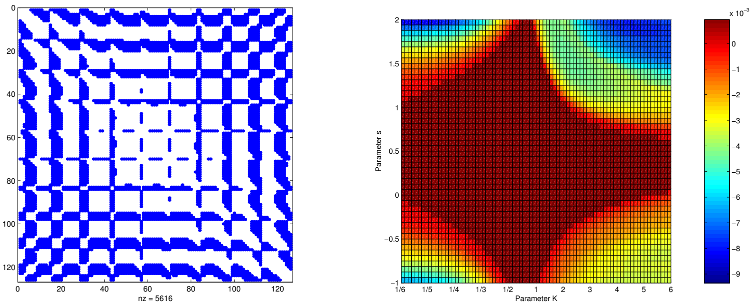

Figure 3: Left: Negative entries in the inverse matrix for $K = 30$ and $s = 1.5$. Right: Dependence of the minimal entry of the inverse stiffness matrix on parameters s and K.

We computed the stiffness matrices for many combinations of parameters K and s, and found the minimal values of entries of their inverses. The colour of each pixel in Fig. 3 (right) corresponds to the minimal value of the inverse stiffness matrix for a given pair of parameters. The vertical axis describes the dependence of the minimum on the slope parameter s. The range overlaps the interval $(0, 1)$, so it shows the area of non-acute triangulations, too.

Condition (14) can be modified to the form

$$\frac{4\,\bar{b}\,\cos\alpha_1}{d\,C^2\,\operatorname{ctg} 2\alpha_1} \leq 1. \tag{19}$$

Values of the expression on the left-hand side of (19) are shown in Fig. 4. By Theorem 1, the DMP is satisfied for those values that are less or equal to one. Note that the functions in Figs. 3 and 4 need not be symmetric, because the shape of $\Omega(s)$ is different for $s = 0$ and $s = 1$.

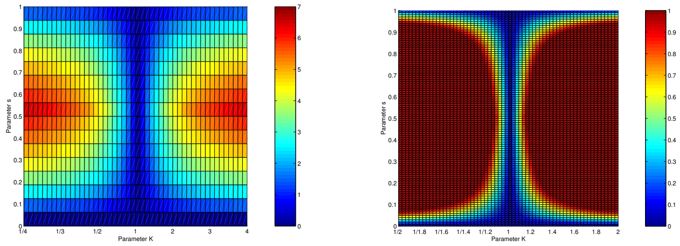

Figure 4: Left: Behaviour of the left-side of condition (19). Right: Areas of the validity of (19). The values larger than 1 are indicated by a dark colour.

Example: Fig. 5 shows that the area, where the DMP is violated, can be in the middle of the domain for $K = 8$ and $s = 0$.

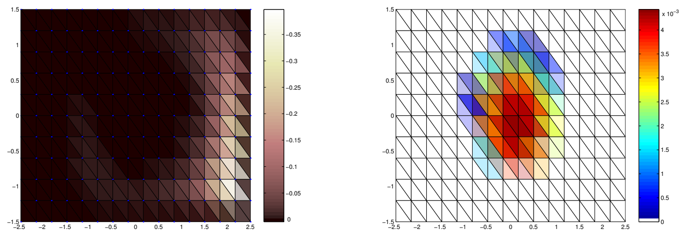

Figure 5: Left: The discrete solution u_h for a certain F. Right: A domain, where DMP is violated.

6 Conclusions and open problems

The above analysis shows that DMP is valid on acute and regular triangulations for full diffusion tensors with a sufficiently strong diagonal dominance property.

Construction of acute triangulations is quite well studied, e.g., in works [14], [17]. The family of uniformly acute triangulations can be easily constructed using the standard $2d$ red refinement of triangles by midlines.

It is worth mentioning that the idea proposed here cannot be easily generalized for elliptic problems in dimension 3 and higher, since it is not clear how to construct and further refine acute simplicial meshes of a given polytopic domain (cf. [3], [10]).

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