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Abstract: *A residual based a posteriori estimator for the reaction-diffusion problem is introduced. We show that the estimator gives both an upper and a lower bound to error. Numerical results are presented.*

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1 Introduction

We consider the finite element approximation of the reaction-diffusion problem

$$-\varepsilon^2 \Delta u + u = f \text{ in } \Omega \quad \text{ and } \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

with the parameter $\varepsilon > 0$. For $\varepsilon \gtrsim 1$ the problem is a standard elliptic equation. We are, however, interested in the case of a "small" $\varepsilon << 1$. In this case, the problem is a singularly perturbed problem, and the question is how to incorporate the effect of ε into the finite element a posteriori analysis. The problem has been studied for example in [4, 1]. Here we introduce and analyze an alternative a posteriori estimator. In [2], this is extended to the Brinkman equations modeling flow in porous media.

2 The a posteriori error estimate

Let $\Omega \subset \mathbb{R}^n$ be a domain with a polygonal or a polyhedral boundary $\partial\Omega$. We assume a shape regular triangular/tetrahedral partitioning $\mathcal{C}_h$ of the domain Ω. With h_K we denote the diameter of $K \in \mathcal{C}_h$ and we let $h = \max h_K$. With $\mathcal{E}_h$ we denote the internal edges (faces in 3D) of $\mathcal{C}_h$. The constant C is a generic constant independent of the mesh size and problem parameter ε.

Defining the bilinear form

$$\mathcal{A}(u, v) = \varepsilon^2 (\nabla u, \nabla v) + (u, v), \tag{2}$$

the weak form of the problem is: find $u \in V$ such that

$$\mathcal{A}(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Defining $V_h = \{v \in H_0^1(\Omega) \mid v_{|K} \in P_k(K) \ \forall K \in \mathcal{C}_h\}$, the finite element method is: find $u_h \in V_h$ such that

$$\mathcal{A}(u_h, v) = (f, v) \quad \forall v \in V_h. \tag{4}$$

The natural energy norm is

$$\|v\|_\varepsilon^2 = \varepsilon^2 \|\nabla v\|_0^2 + \|v\|_0^2, \tag{5}$$

and the finite element solution is the best approximation with respect to this norm

$$\|u - u_h\|_\varepsilon = \inf_{v \in V_h} \|u - v\|_\varepsilon. \tag{6}$$

In general, the problem has a boundary layer of the form $e^{-d/\varepsilon}$, where d is the distance from the boundary. Hence, even for a smooth load f, a uniform mesh will only lead to the following estimate

$$\|u - u_h\|_\varepsilon \le C\sqrt{h} \tag{7}$$

uniformly valid with respect to ε. For a smooth solution the estimate obtained is

$$\|u - u_h\|_\varepsilon \leq C(\varepsilon h^k + h^{k+1}). \tag{8}$$

To improve the convergence, adaptive mesh refinement is natural. Here, we we introduce a novel residual based a posteriori estimator. The elementwise estimator is defined as

$$E_K(u_h)^2 = \frac{h_K^2}{\varepsilon^2 + h_K^2}\|\varepsilon^2 \Delta u_h - u_h + f\|_{0,K}^2 + \frac{h_K}{\varepsilon^2 + h_K^2}\|[\![\varepsilon^2 \partial_n u_h]\!]\|_{0,\partial K \cap \mathcal{E}_h}^2 \tag{9}$$

and the global estimator is

$$\eta = \Big(\sum_{K \in \mathcal{C}_h} E_K(u_h)^2\Big)^{1/2}. \tag{10}$$

Above $[\![\cdot]\!]$ denotes the jump and ∂_n denotes the normal derivative.

If $\varepsilon \gtrsim 1$, the elementwise estimator recovers the usual estimator for second order elliptic equations

$$E_K(u_h)^2 \approx h_K^2\|\varepsilon^2 \Delta u_h - u_h + f\|_{0,K}^2 + h_K\|[\![\varepsilon^2 \partial_n u_h]\!]\|_{0,\partial K \cap \mathcal{E}_h}^2.$$

On the other hand, in the limit $\varepsilon \to 0$ (or $\varepsilon \ll h$), when the FE solution is the L^2-projection of the loading, we have $E_K(u_h)^2 \approx \|-u_h + f\|_{0,K}^2$.

For our analysis we will need a saturation assumption. The partitioning $\mathcal{C}_h$ is refined into $\mathcal{C}_{h/2}$ by dividing each triangle/tetrahedron K into four/eight elements with mesh size $h_K/2$. By $u_{h/2} \in V_{h/2}$ we denote the finite element solution on the refined mesh.

Assumption 1. *There exists a positive constant $\beta < 1$ such that*

$$\|u - u_{h/2}\|_\varepsilon \leq \beta\|u - u_h\|_\varepsilon. \tag{11}$$

The main result is the following theorem.

Theorem 2. *Let Assumption 1 hold. Then there exists $C > 0$ such that*

$$\|u - u_h\|_\varepsilon \leq C\eta. \tag{12}$$

Proof. By the triangle inequality the saturation assumption gives

$$\|u - u_h\|_\varepsilon \leq \frac{C}{1-\beta}\big(\|u_{h/2} - u_h\|_\varepsilon\big). \tag{13}$$

Next, with $v = (u_{h/2} - u_h)/\|u_{h/2} - u_h\|_\varepsilon$, we have

$$\|u_{h/2} - u_h\|_\varepsilon = \mathcal{A}(u_{h/2} - u_h, v) \tag{14}$$

and $\|v\|_\varepsilon = 1$. Let $\tilde{v} \in V_h$ be the Lagrange interpolant of v. Since both v and $\tilde{v}$ are in the finite element spaces, scaling arguments give

$$\begin{aligned}\Big(\sum_{K \in \mathcal{C}_{h/2}} \Big(\frac{\varepsilon + h_K}{h_K}\Big)^2 \|v - \tilde{v}\|_{0,K}^2\Big)^{1/2} \\ \leq C\Big(\sum_{K \in \mathcal{C}_{h/2}} \big(\varepsilon^2\|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2\big)\Big)^{1/2} = C\|v\|_\varepsilon = C\end{aligned} \tag{15}$$

and

$$
\Big(\sum_{K\in\mathcal{C}_{h/2}} \frac{\varepsilon^2+h_K^2}{h_K}\|v-\tilde v\|_{0,\partial K}^2\Big)^{1/2} \le C\Big(\sum_{K\in\mathcal{C}_{h/2}} \frac{\varepsilon^2+h_K^2}{h_K}\, h_K^{-1}\|v-\tilde v\|_{0,K}^2\Big)^{1/2}
$$

$$
= C\Big(\sum_{K\in\mathcal{C}_{h/2}} \big(\frac{\varepsilon^2}{h_K^2}+1\big)\|v-\tilde v\|_{0,K}^2\Big)^{1/2} \le C\Big(\sum_{K\in\mathcal{C}_{h/2}} \big(\varepsilon^2\|\nabla v\|_{0,K}^2+\|v\|_{0,K}^2\big)\Big)^{1/2}
$$

$$
= C\|v\|_\varepsilon = C. \tag{16}
$$

Since it holds $\mathcal{A}(u_{h/2}-u_h,\tilde v)=0$, we have

$$
\mathcal{A}(u_{h/2}-u_h,v)=\mathcal{A}(u_{h/2}-u_h,v-\tilde v). \tag{17}
$$

Using the fact that $u_{h/2}$ satisfies

$$
\mathcal{A}(u_{h/2},v-\tilde v)=(f,v-\tilde v) \tag{18}
$$

and integrating by parts, we get

$$
\begin{aligned}
&\mathcal{A}(u_{h/2}-u_h,v-\tilde v)\\
&=(f,v-\tilde v)-\varepsilon^2\left(\nabla u_h,\nabla(v-\tilde v)\right)-(u_h,v-\tilde v)\\
&=\sum_{K\in\mathcal{C}_{h/2}}\Big\{\left(\varepsilon^2\Delta u_h-u_h+f,v-\tilde v\right)_K+\varepsilon^2\left\langle\partial_n u_h,v-\tilde v\right\rangle_{\partial K\cap\mathcal{E}_{h/2}}\Big\}.
\end{aligned} \tag{19}
$$

Using Schwartz inequality and the estimates (15)–(16) we then obtain

$$
\mathcal{A}(u_{h/2}-u_h,v-\tilde v)\le C\eta. \tag{20}
$$

□

The a posteriori upper bound η is also a lower bound to the error. In this sense the estimator is sharp. The proof of the following theorem uses classical techniques, see [3].

Theorem 3. *Let $f_h\in V_h$ an approximation of the load f. Then there exist $C>0$ such that*

$$
\eta^2\le C\Big\{\|u-u_h\|_\varepsilon^2+\sum_{K\in\mathcal{C}_h}\big(\frac{h_K^2}{\varepsilon^2+h_K^2}\|f-f_h\|_{0,K}^2\big)\Big\}. \tag{21}
$$

3 Numerical results

For the computations we choose the unit square $\Omega=(0,1)\times(0,1)$ and a unit load $f=1$. For the number of degrees of freedom N, the uniform estimate (7) and the asymptotic estimate (8) become

$$
\|u-u_h\|_\varepsilon\le CN^{-0.25}\qquad\text{and}\qquad\|u-u_h\|_\varepsilon\le CN^{-k/2}, \tag{22}
$$

respectively. In Figure 1 this behavior is seen for linear and quadratic elements ($k=1,2$).

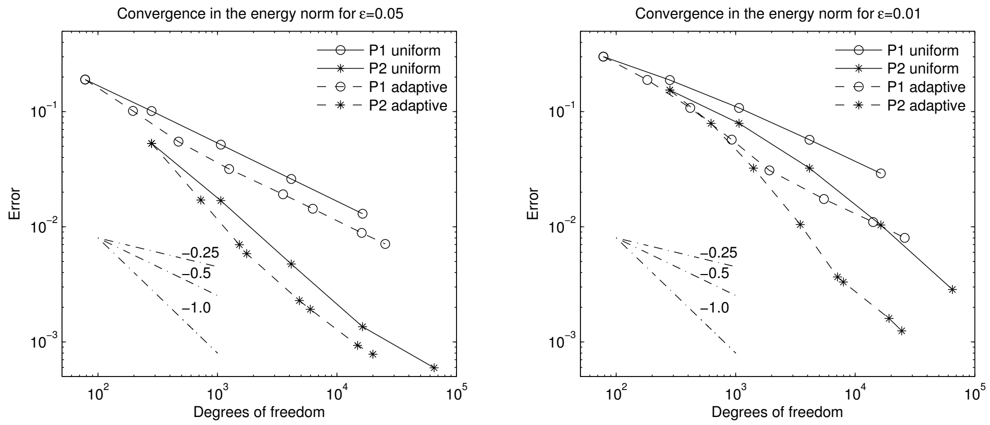

Figure 1: Convergence for uniform and adaptive meshes for parameter values $\varepsilon = 0.05$ and $\varepsilon = 0.01$.

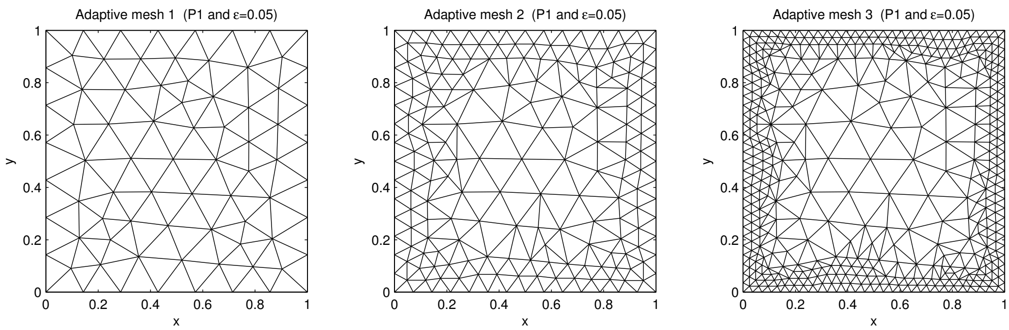

Figure 2: First three meshes of the adaptive scheme using linear elements and parameter value $\epsilon = 0.05$.

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