

A RESIDUAL BASED A POSTERIORI ESTIMATOR FOR THE REACTION–DIFFUSION PROBLEM

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Abstract: *A residual based a posteriori estimator for the reaction-diffusion problem is introduced. We show that the estimator gives both an upper and a lower bound to error. Numerical results are presented.*

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1 Introduction

We consider the finite element approximation of the reaction-diffusion problem

$$-\varepsilon^2 \Delta u + u = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

with the parameter $\varepsilon > 0$. For $\varepsilon \gtrsim 1$ the problem is a standard elliptic equation. We are, however, interested in the case of a "small" $\varepsilon \ll 1$. In this case, the problem is a singularly perturbed problem, and the question is how to incorporate the effect of ε into the finite element a posteriori analysis. The problem has been studied for example in [4, 1]. Here we introduce and analyze an alternative a posteriori estimator. In [2], this is extended to the Brinkman equations modeling flow in porous media.

2 The a posteriori error estimate

Let $\Omega \subset \mathbb{R}^n$ be a domain with a polygonal or a polyhedral boundary $\partial\Omega$. We assume a shape regular triangular/tetrahedral partitioning \mathcal{C}_h of the domain Ω . With h_K we denote the diameter of $K \in \mathcal{C}_h$ and we let $h = \max h_K$. With \mathcal{E}_h we denote the internal edges (faces in 3D) of \mathcal{C}_h . The constant C is a generic constant independent of the mesh size and problem parameter ε .

Defining the bilinear form

$$\mathcal{A}(u, v) = \varepsilon^2 (\nabla u, \nabla v) + (u, v), \quad (2)$$

the weak form of the problem is: find $u \in V$ such that

$$\mathcal{A}(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (3)$$

Defining $V_h = \{v \in H_0^1(\Omega) \mid v|_K \in P_k(K) \ \forall K \in \mathcal{C}_h\}$, the finite element method is: find $u_h \in V_h$ such that

$$\mathcal{A}(u_h, v) = (f, v) \quad \forall v \in V_h. \quad (4)$$

The natural energy norm is

$$\|v\|_\varepsilon^2 = \varepsilon^2 \|\nabla v\|_0^2 + \|v\|_0^2, \quad (5)$$

and the finite element solution is the best approximation with respect to this norm

$$\|u - u_h\|_\varepsilon = \inf_{v \in V_h} \|u - v\|_\varepsilon. \quad (6)$$

In general, the problem has a boundary layer of the form $e^{-d/\varepsilon}$, where d is the distance from the boundary. Hence, even for a smooth load f , a uniform mesh will only lead to the following estimate

$$\|u - u_h\|_\varepsilon \leq C\sqrt{h} \quad (7)$$

uniformly valid with respect to ε . For a smooth solution the estimate obtained is

$$\|u - u_h\|_\varepsilon \leq C(\varepsilon h^k + h^{k+1}). \quad (8)$$

To improve the convergence, adaptive mesh refinement is natural. Here, we introduce a novel residual based a posteriori estimator. The elementwise estimator is defined as

$$E_K(u_h)^2 = \frac{h_K^2}{\varepsilon^2 + h_K^2} \|\varepsilon^2 \Delta u_h - u_h + f\|_{0,K}^2 + \frac{h_K}{\varepsilon^2 + h_K^2} \|\llbracket \varepsilon^2 \partial_n u_h \rrbracket\|_{0,\partial K \cap \mathcal{E}_h}^2 \quad (9)$$

and the global estimator is

$$\eta = \left(\sum_{K \in \mathcal{C}_h} E_K(u_h)^2 \right)^{1/2}. \quad (10)$$

Above $\llbracket \cdot \rrbracket$ denotes the jump and ∂_n denotes the normal derivative.

If $\varepsilon \gtrsim 1$, the elementwise estimator recovers the usual estimator for second order elliptic equations

$$E_K(u_h)^2 \approx h_K^2 \|\varepsilon^2 \Delta u_h - u_h + f\|_{0,K}^2 + h_K \|\llbracket \varepsilon^2 \partial_n u_h \rrbracket\|_{0,\partial K \cap \mathcal{E}_h}^2.$$

On the other hand, in the limit $\varepsilon \rightarrow 0$ (or $\varepsilon \ll h$), when the FE solution is the L^2 -projection of the loading, we have $E_K(u_h)^2 \approx \| -u_h + f \|_{0,K}^2$.

For our analysis we will need a saturation assumption. The partitioning \mathcal{C}_h is refined into $\mathcal{C}_{h/2}$ by dividing each triangle/tetrahedron K into four/eight elements with mesh size $h_K/2$. By $u_{h/2} \in V_{h/2}$ we denote the finite element solution on the refined mesh.

Assumption 1. *There exists a positive constant $\beta < 1$ such that*

$$\|u - u_{h/2}\|_\varepsilon \leq \beta \|u - u_h\|_\varepsilon. \quad (11)$$

The main result is the following theorem.

Theorem 2. *Let Assumption 1 hold. Then there exists $C > 0$ such that*

$$\|u - u_h\|_\varepsilon \leq C\eta. \quad (12)$$

Proof. By the triangle inequality the saturation assumption gives

$$\|u - u_h\|_\varepsilon \leq \frac{C}{1 - \beta} (\|u_{h/2} - u_h\|_\varepsilon). \quad (13)$$

Next, with $v = (u_{h/2} - u_h)/\|u_{h/2} - u_h\|_\varepsilon$, we have

$$\|u_{h/2} - u_h\|_\varepsilon = \mathcal{A}(u_{h/2} - u_h, v) \quad (14)$$

and $\|v\|_\varepsilon = 1$. Let $\tilde{v} \in V_h$ be the Lagrange interpolant of v . Since both v and \tilde{v} are in the finite element spaces, scaling arguments give

$$\begin{aligned} & \left(\sum_{K \in \mathcal{C}_{h/2}} \left(\frac{\varepsilon + h_K}{h_K} \right)^2 \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2} \\ & \leq C \left(\sum_{K \in \mathcal{C}_{h/2}} (\varepsilon^2 \|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2) \right)^{1/2} = C \|v\|_\varepsilon = C \end{aligned} \quad (15)$$

and

$$\begin{aligned}
& \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K} \|v - \tilde{v}\|_{0,\partial K}^2 \right)^{1/2} \leq C \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K} h_K^{-1} \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2} \\
& = C \left(\sum_{K \in \mathcal{C}_{h/2}} \left(\frac{\varepsilon^2}{h_K^2} + 1 \right) \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2} \leq C \left(\sum_{K \in \mathcal{C}_{h/2}} (\varepsilon^2 \|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2) \right)^{1/2} \\
& = C \|v\|_\varepsilon = C. \tag{16}
\end{aligned}$$

Since it holds $\mathcal{A}(u_{h/2} - u_h, \tilde{v}) = 0$, we have

$$\mathcal{A}(u_{h/2} - u_h, v) = \mathcal{A}(u_{h/2} - u_h, v - \tilde{v}). \tag{17}$$

Using the fact that $u_{h/2}$ satisfies

$$\mathcal{A}(u_{h/2}, v - \tilde{v}) = (f, v - \tilde{v}) \tag{18}$$

and integrating by parts, we get

$$\begin{aligned}
& \mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) \\
& = (f, v - \tilde{v}) - \varepsilon^2 (\nabla u_h, \nabla(v - \tilde{v})) - (u_h, v - \tilde{v}) \\
& = \sum_{K \in \mathcal{C}_{h/2}} \left\{ (\varepsilon^2 \Delta u_h - u_h + f, v - \tilde{v})_K + \varepsilon^2 \langle \partial_n u_h, v - \tilde{v} \rangle_{\partial K \cap \mathcal{E}_{h/2}} \right\}. \tag{19}
\end{aligned}$$

Using Schwartz inequality and the estimates (15)–(16) we then obtain

$$\mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) \leq C\eta. \tag{20}$$

□

The a posteriori upper bound η is also a lower bound to the error. In this sense the estimator is sharp. The proof of the following theorem uses classical techniques, see [3].

Theorem 3. *Let $f_h \in V_h$ an approximation of the load f . Then there exist $C > 0$ such that*

$$\eta^2 \leq C \left\{ \|u - u_h\|_\varepsilon^2 + \sum_{K \in \mathcal{C}_h} \left(\frac{h_K^2}{\varepsilon^2 + h_K^2} \|f - f_h\|_{0,K}^2 \right) \right\}. \tag{21}$$

3 Numerical results

For the computations we choose the unit square $\Omega = (0, 1) \times (0, 1)$ and a unit load $f = 1$. For the number of degrees of freedom N , the uniform estimate (7) and the asymptotic estimate (8) become

$$\|u - u_h\|_\varepsilon \leq CN^{-0.25} \quad \text{and} \quad \|u - u_h\|_\varepsilon \leq CN^{-k/2}, \tag{22}$$

respectively. In Figure 1 this behavior is seen for linear and quadratic elements ($k = 1, 2$).

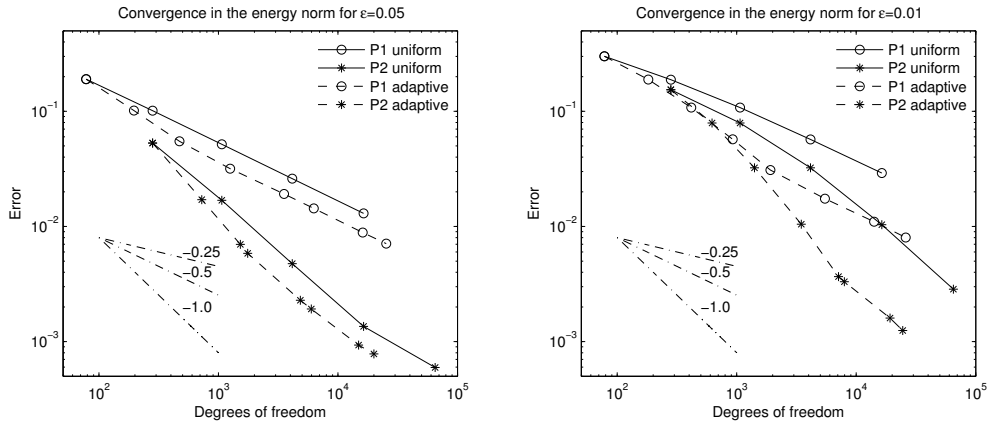


Figure 1: Convergence for uniform and adaptive meshes for parameter values $\epsilon = 0.05$ and $\epsilon = 0.01$.

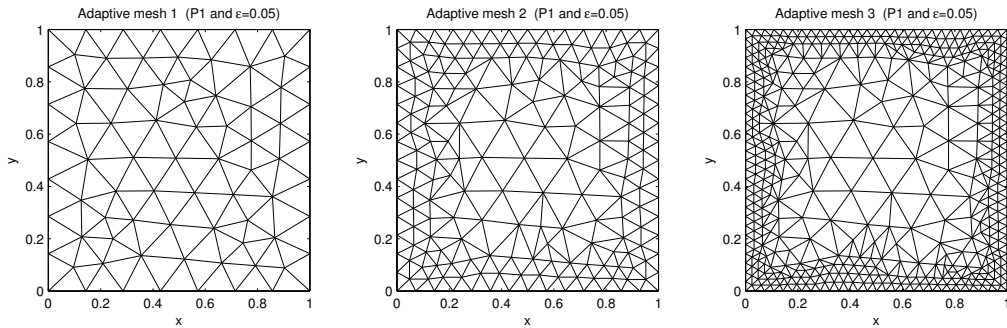


Figure 2: First three meshes of the adaptive scheme using linear elements and parameter value $\epsilon = 0.05$.

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