Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2009 A576

ON ZEROS AND BOUNDARY BEHAVIOR OF BOUNDED HARMONIC FUNCTIONS

S. Ponnusamy Antti Rasila

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2009 A576

ON ZEROS AND BOUNDARY BEHAVIOR OF BOUNDED HARMONIC FUNCTIONS

S. Ponnusamy **Antti Rasila**

Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis

S. Ponnusamy, Antti Rasila: *On zeros and boundary behavior of bounded harmonic functions*; Helsinki University of Technology Institute of Mathematics Research Reports A576 (2009).

Abstract: *We study the connection between multiplicities of the zeros and boundary behavior of bounded analytic and harmonic functions. We prove existence of angular (non-tangential) limit at a boundary point provided that multiplicities of zeroes of the function grow fast enough on a given sequence of points approaching the boundary.*

AMS subject classifications: 30C15, 31A20

Keywords: analytic and planar harmonic functions, angular limits, multiplicity of zeroes

Correspondence

S. Ponnusamy
Indian Institute of Technology Madras
Department of Mathematics
Chennai 600 036
India

Antti Rasila
Helsinki University of Technology
Department of Mathematics and Systems Analysis
P.O. Box 1100
FI-02015 TKK
Finland

samy@iitm.ac.in, antti.rasila@tkk.fi

ISBN 978-952-248-043-9 (print)
ISBN 978-952-248-044-6 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)

Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis
P.O. Box 1100, FI-02015 TKK, Finland
email: math@tkk.fi http://math.tkk.fi/

1 Introduction

A function $f(z)$ is said to have a Lindelöf property in the unit disk $\mathbb{D}$ if whenever $f(z) \to \alpha$ as $z \to z_0 \in \partial\mathbb{D}$ along some arc lying in $\mathbb{D}$ and terminating at z_0, then $f(z) \to \alpha$ uniformly as $z \to z_0$ inside any angular domain of opening $\pi - \epsilon$ in $\mathbb{D}$ with z_0 as its vertex which is bisected by the radius drawn to z_0. In this case, $f(z)$ has the *angular limit* α at z_0.

The following classical result is known as Lindelöf's theorem (see e.g. [12, p.259]):

1.1 Theorem. *Suppose that γ is a curve, with parametric interval $[0,1]$, such that $|\gamma(t)| < 1$ if $t < 1$ and $\gamma(1) = 1$. If f is a bounded analytic function of the unit disk $\mathbb{D}$ and*

$$\lim_{t\to 1} f\big(\gamma(t)\big) = \alpha,$$

then f has angular limit α at 1.

There are various generalizations of Lindelöf's theorem in the literature. It is interesting to ask whether a weaker condition would be sufficient for the result. Another related result is due to P. Koebe. He proved that if a bounded analytic function tends to zero along a sequence of arcs in the unit disk which approaches a subarc in the boundary, and if the Euclidean diameters of these arcs are bounded from below by a constant $c > 0$, then it must be identically zero [4, p.19]. A recent survey on the results of this type is given in [7]. For results concerning sequential limits, see also [2].

This topic has been studied by several authors, in particular by Rung, who studied the connection between the boundary behavior of analytic functions and the hyperbolic metric in [13]. In Rung's results, the values attained by the function are assumed to approach a limit at a certain rate on a sequence of continua of given hyperbolic diameter. By studying the balance between the rate of convergence and the growth of the hyperbolic diameter, one can make conclusions on the limit behavior of the function. For example, Rung proved the following result:

1.2 Theorem. *[13, Corollary 1] Suppose that γ is a boundary path in the unit disk $\mathbb{D}$, and f is analytic in $\mathbb{D}$ such that for some w_0 and for some positive function $A(r)$, $r \in [0,1)$,*

$$\log|f(z) - w_0| \le \frac{-A(r)}{(1-|z|)},$$

for $z \in |\gamma|$, $|z| \ge r$ and

$$\liminf_{r\to 1} \frac{M(r,f)}{A(r)} = 0,$$

then $f(z) \equiv w_0$. Here $M(r,f) = \max\{\sup_{|z|<r} \log|f(z)|, 1\}$.

In this paper we show results of above types under assumptions involving the multiplicities of zeroes of the function.

2 Preliminaries

Let $f(z)$ be analytic at z_0 and suppose that $f(z_0) = 0$, but $f(z)$ is not identically zero. Then $f(z)$ is said to have a zero of order n at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0, \text{ and } f^{(n)}(z_0) \neq 0.$$

If $f(z)$ is analytic at z_0, and has zero of order n at z_0, we write $\mu(z_0, f) = n$. We first recall the following well-known version of the classical Schwarz' lemma.

2.1 Lemma. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$ and $\mu(0, f) = p \geq 1$. Then*

$$|f(z)| \leq |z|^p \textit{ for all } z \in \mathbb{D}.$$

The hyperbolic metrics in the upper half plane $\mathbb{H}$ and in the unit disk $\mathbb{D}$ are defined by

$$\cosh \rho_{\mathbb{H}}(z, w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}\,(z)\,\mathrm{Im}\,(w)}, \quad z, w \in \mathbb{H}, \tag{2.2}$$

and

$$\sinh^2\Big(\frac{1}{2}\rho_{\mathbb{D}}(z, w)\Big) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}, \quad z, w \in \mathbb{D}, \tag{2.3}$$

respectively. If there is no danger of confusion, we denote both $\rho_{\mathbb{H}}(z, w)$ and $\rho_{\mathbb{D}}(z, w)$ simply by $\rho(z, w)$. A hyperbolic disk with the center z and the radius $M > 0$ is denoted by $D(z, M)$.

Both for $(\mathbb{D}, \rho_{\mathbb{D}})$ and $(\mathbb{H}, \rho_{\mathbb{H}})$ one can define the hyperbolic distance in terms of the absolute ratio. Since the absolute ratio is invariant under Möbius transformations, the hyperbolic metric also remains invariant under these transformations. The proof will use the hyperbolic form of Pythagoras' Theorem.

2.4 Lemma. *[3, Theorem 7.11.1] For a hyperbolic triangle with angles $\alpha, \beta, \pi/2$ and corresponding hyperbolic opposite side lengths a, b, c, we have*

$$\cosh c = \cosh a \cosh b.$$

We have the following formula for $z \in \mathbb{D}$:

$$\rho_{\mathbb{D}}(0, z) = \log \frac{1+|z|}{1-|z|}. \tag{2.5}$$

This relation immediately yields that

$$|z| = \tanh\Big(\frac{1}{2}\rho_{\mathbb{D}}(0, z)\Big) < 1, \tag{2.6}$$

where we have used the fact that

$$\tanh z = \frac{e^{2z}-1}{e^{2z}+1}.$$

Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic. Then the following inequality holds for the hyperbolic distance (see for example [8, p. 268])

$$\rho_{\mathbb{D}}\big(f(x), f(y)\big) \leq \rho_{\mathbb{D}}(x, y) \text{ for } x, y \in \mathbb{D}, \tag{2.7}$$

where the equality holds if and only if f is a Möbius transformation.

3 Bounded analytic functions

We will show a connection between the multiplicity of the zeros and the boundary behavior of bounded analytic functions. A similar method of proof was used in the context of bounded quasiregular mappings of $\mathbb{R}^n$, $n \geq 2$ in [11].

3.1 Theorem. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to \beta \in \partial\mathbb{D}$, $\rho_{\mathbb{D}}(b_k, b_{k+1}) \leq M_k < \infty$ and $\mu_k = \mu(b_k, f)$. If*

$$\left(\frac{1 - e^{-M_k}}{1 + e^{-M_k}}\right)^{\mu_k} \to 0 \ \textit{as} \ k \to \infty, \tag{3.2}$$

then $f(z)$ has an angular limit 0 at β.

Proof. Let $\rho_{\mathbb{D}}(b_k, b_{k+1}) \leq M_k$. We may choose a Möbius transformation h_k such that $h_k(\mathbb{D}) = \mathbb{D}$ and $h_k(0) = b_k$. Then the function $g_k = f \circ h_k$ is analytic, $g_k\colon \mathbb{D} \to \mathbb{D}$ and $g_k(0) = 0$. Inequality (2.7), applied to h_k^{-1} with $x = b_k$, $y = b_{k+1}$ gives

$$\rho_{\mathbb{D}}\big(0, h_k^{-1}(b_{k+1})\big) \leq M_k.$$

Let $R_k = \tanh(\frac{1}{2}M_k)$ and $\mu_k = \mu(b_k, f)$. We see that g_k has a zero of order μ_k at the origin. It follows by Lemma 2.1 and (2.6) that, for $|z| \leq R_k$,

$$|g_k(z)| \leq |z|^{\mu_k} \leq R_k^{\mu_k}$$

or equivalently $\rho_{\mathbb{D}}(0, z) \leq M_k$. Thus

$$|f(z)| \leq R_k^{\mu_k} \text{ for } z \in [b_k, b_{k+1}], \tag{3.3}$$

where $[b_k, b_{k+1}]$ is the line segment connecting the points b_k and b_{k+1}. Let γ be the broken line joining the points b_k and b_{k+1} for $k = 1, 2, \ldots$. Because

$$\left(\frac{1 - e^{-M_k}}{1 + e^{-M_k}}\right)^{\mu_k} \to 0 \text{ as } k \to \infty,$$

we have

$$\mu_k \log \frac{1 + e^{-M_k}}{1 - e^{-M_k}} \to \infty \text{ as } k \to \infty. \tag{3.4}$$

From (2.5) we obtain

$$e^{-M_k} = \frac{1 - R_k}{1 + R_k},$$

and hence

$$\frac{1 + e^{-M_k}}{1 - e^{-M_k}} = \frac{1}{R_k}.$$

This observation together with (3.4) yields

$$\mu_k \log R_k \to -\infty \text{ or } R_k^{\mu_k} \to 0,$$

as $k \to \infty$. It follows by (3.3) that f has a limit along γ, and, by Theorem 1.1, f has an angular limit 0 at β. $\square$

3.5 Corollary. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to \beta \in \partial\mathbb{D}$, $\rho_{\mathbb{D}}(b_k, b_{k+1}) \le M < \infty$ and $\mu(b_k, f) \to \infty$. Then $f(z)$ has an angular limit 0 at β.*

3.6 Example. Let $b_k = 1 - 2^{-k}$ and $\mu_k = k$ for $k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{\infty} \mu_k(1 - b_k) = \sum_{k=1}^{\infty} k2^{-k} = 2 < \infty,$$

and hence by [5, Theorem 2.4], one may construct an analytic function $B(z)$ whose zeros are precisely $\{b_k\}$ with respective multiplicities μ_k. It follows from Corollary 3.5 that $B(z)$ has angular limit 0 at 1.

3.7 Remark. It is possible to construct a Blaschke product $B_0\colon \mathbb{D} \to \mathbb{D}$ with infinitely many zeroes b_k on the positive real axis such that $b_k \to 1$ and $\mu(b_k, B_0) \to \infty$ as $k \to \infty$, but B_0 does not have an angular limit at 1. A construction due to P. Lappan is given in [11, 5.21].

3.8 Theorem. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to 1$ and $\mu_k = \mu(b_k, f)$. If*

$$\liminf_{k\to\infty} |b_k|^{\mu_k} = 0,$$

then $f \equiv 0$ on $\mathbb{D}$.

Proof. Let h_k be a Möbius transformation such that $h_k(\mathbb{D}) = \mathbb{D}$, $h_k(0) = b_k$, and $g_k = f \circ h_k$, as before. By Lemma 2.1, we have

$$|g_k(z)| \le |b_k|^{\mu_k} \text{ for } |z| \le |b_k|,$$

and hence

$$|f(z)| \le R^{\mu_k} \text{ for } z \in D(b_k, M_k),$$

where

$$M_k = \rho_{\mathbb{D}}(0, |b_k|) = \frac{1}{2}\log\frac{1 + |b_k|}{1 - |b_k|}.$$

Because $b_k \to 1$ as $k \to \infty$, the line segment $(0, 1/2) \subset D(b_k, M_k)$ for sufficiently large values of k. It follows that if

$$\liminf_{k\to\infty} |b_k|^{\mu_k} = 0,$$

then $f(z) = 0$ for all $z \in (0, 1/2)$ and hence, by the uniqueness theorem, f is identically zero on $\mathbb{D}$. $\square$

4 Harmonic case

In this section we give a harmonic analog of Theorem 3.1. Let $\Omega \subset \mathbb{C}$ be a simply connected domain. A complex-valued function $f\colon \Omega \to \mathbb{C}$, $(x, y) \mapsto (u, v)$, is *planar harmonic* if the two coordinate functions u and v are (real) harmonic in Ω. It is well-known that $f\colon \Omega \to \mathbb{C}$ is a planar harmonic function if and only if the function f has the representation $f = h + \overline{g}$, where h and g are analytic in Ω. The representation is unique up to an additive constant. We call the functions h and g the analytic and the co-analytic parts of f, respectively. For basic properties of planar harmonic functions we refer to the monograph [6].

4.1 Example. The bounded harmonic mapping $f\colon \mathbb{D} \to \mathbb{C}$ defined by

$$f(z) = \mathrm{Arg}\,(1 - z) + i\mathrm{Re}\,(1 - z)$$

has infinitely many asymptotic values at 1. Hence it is clear that Lindelöf's theorem does not generalize to the harmonic functions.

But having the same limit when approaching a boundary point radially from two separate directions is sufficient for an angular limit.

4.2 Theorem. *[1, Theorem 2.10] Suppose that $u\colon \mathbb{H} \to \mathbb{R}$ is bounded and harmonic on $\mathbb{H}$. If $0 < \theta_1 < \theta_2 < \pi$ and*

$$\lim_{r\to 0} u(re^{i\theta_1}) = L = \lim_{r\to 0} u(re^{i\theta_2}),$$

then u has an angular limit L at 0.

In order to recall our next result, we need to introduce the definition of the multiplicity for sense-preserving harmonic function f in $\mathbb{D}$ which has the decomposition of the form $f = h + \overline{g}$. A complex-valued harmonic function f is sense-preserving in $\mathbb{D}$ if it satisfies a Beltrami equation $\overline{f_{\bar{z}}} = \omega f_z$, where ω is an analytic function in $\mathbb{D}$ with $|\omega(z)| < 1$. Suppose that h and g have respectively multiplicity n and m at 0 with $n \le m$. Then $f = h\psi$, where $\psi = 1 + \chi$ and $\chi = \overline{g}/h$. Without loss of generality, we can suppose that $\psi(z) \neq 0$ in $\mathbb{D}$ and $h(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. Then we say that f has zero of order n at z_0 and write $\mu(z_0, f) = n$.

For a positively oriented circle γ with center at the origin, in $\mathbb{D}$, we have

$$\Delta_\gamma \arg f = \Delta_\gamma \arg h + \Delta_\gamma \arg \psi$$

where $\Delta_\gamma \arg f$ defines the change in $\arg f$ as z traverse around γ. Since $\Delta_\gamma \arg \psi = 0$ and $\Delta_\gamma \arg h = 2\,\pi n$, it follows that $\Delta_\gamma \arg h = 2\pi n$. Consequently, f has at least n zeros in $\mathbb{D}$ (with counting multiplicity). Since $J_f > 0$, f cannot have more than n zeros at the origin in $\mathbb{D}$ counting multiplicity as a positive number. Hence, f has at most n zeros in $\mathbb{D}$ (with counting multiplicity). If $m > n$, then $\phi = g'/h' = cz^r$, where $r = m - n > 0$, and so, J_f is positive in $N \setminus \{0\}$, where N is a neighborhood of 0, and consequently, f is locally univalent. See also [6, 9].

The following lemma is due to Mateljević and Vuorinen [10] from an unpublished manuscript. In view of this, we outline the proof of this result.

4.3 Lemma. *Let f be a sense-preserving harmonic function of $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|^{\mu(0,f)} \leq \frac{4}{\pi} |z|^{\mu(0,f)} \quad \textit{for } z \in \mathbb{D}. \tag{4.4}$$

Proof. Let $\phi(z) = \frac{4}{\pi} \tan^{-1}(z)$ and $S = \{z : |\mathrm{Re}\, z| < 1\}$. Suppose that F is an analytic function of $\mathbb{D}$ into S with $F(0) = 0$ and $\mu(0, F) = p \geq 1$. Then $\omega = \phi^{-1} \circ F$ satisfies the conditions of Lemma 2.1 and therefore $|\omega(z)| \leq |z|^p$. Because $F = \phi \circ \omega$ and $|\mathrm{Re}\, \phi(\omega(z))| \leq \frac{4}{\pi}(|\omega(z)|)$, we have

$$|\mathrm{Re}\, F(z)| \leq \tan^{-1}(|z|^p). \tag{4.5}$$

Fix θ and $z \in \mathbb{D}$, and define $H = e^{i\theta} f$. Then we that $H = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. Let $F = h + g$. It follows that $\mathrm{Re}\, H = \mathrm{Re}\, F$. Since $\mu(0, h) \geq p$ and $\mu(0, g) \geq p$, it follows that $\mu(0, F) \geq p$. Now the claim follows from (4.5) and the fact that this inequality holds for all θ. □

4.6 Lemma. *Suppose that $b_1, b_2, \ldots$ is a sequence of points on the positive imaginary axis with $\lim_{k\to\infty} b_k = 0$ and $0 < m < \rho_{\mathbb{H}}(b_k, b_{k+1}) < M_k$. Then there exists $\varphi = \varphi(m)$ such that the angular region*

$$C_\varphi = \{z \in \mathbb{H} : |\arg z - \pi/2| < \varphi \textit{ and } |z| < |b_1|\}$$

is contained in the set $D = \bigcup_{k=1}^{\infty} D(b_k, M_k)$.

Proof. Fix $k \geq 1$. Suppose that $z = re^{i\theta}$, $r > 0$, $\rho(z, b_k) \leq m/2$ and $|\theta - \pi/2| < \varphi_0$, where

$$\varphi_0 = \cos^{-1}\left(\frac{\cosh(m/2)}{\cosh(m)}\right).$$

It is sufficient to show that $z \in D_k = D(b_k, M_k)$.

Consider the hyperbolic triangle with vertices in b_k, ir and z. Denote the sides connecting b_k, ir by a; ir, z by b; and z, b_k by c. Then, by the hyperbolic Pythagoras' Theorem 2.4, we have

$$\cosh(c) = \cosh(a)\cosh(b).$$

It follows that

$$\cosh(b) \leq \frac{\cosh(m)}{\cosh(m/2)}.$$

By [3, (7.20.3)], we have

$$\theta \leq \cos^{-1}\left(\frac{\cosh(m/2)}{\cosh(m)}\right),$$

which is true because $\theta < \varphi_0$. Choosing $\varphi = \varphi_0$ the claim follows. □

4.7 Theorem. *Fix $m > 0$. Let $f\colon \mathbb{H} \to \mathbb{D}$ be a harmonic function, $\mu_k = \mu(b_k, f)$, and let $\{b_k\}$ be a sequence of points on the positive imaginary axis such that $0 < m \le \rho_{\mathbb{H}}(b_k, b_{k+1}) = M_k$ and $f(b_k) = 0$ for all $k = 1, 2, \ldots$ with $\lim_{k\to\infty} b_k = 0$. If*

$$\left(\frac{1-e^{-M_k}}{1+e^{-M_k}}\right)^{\mu_k} \to 0 \textit{ as } k \to \infty,$$

then f has an angular limit 0 at 1.

Proof. By Theorem 4.2, it suffices to show that f has a limit at 0 in the angular region C_φ for some $\varphi \in (0, \pi/2)$.

Let $g_k = f \circ h_k$, where h_k is a Möbius transformation with $h_k(\mathbb{D}) = \mathbb{H}$ and $h_k(0) = b_k$. Then, by Lemma 4.3,

$$|g_k(z)| \le \frac{4}{\pi} \tan^{-1}(R_k^{\mu_k})$$

for $|z| \le R_k$, where $R_k = \tanh(\frac{1}{2}M_k)$. It follows that

$$|f(z)| \le \tan^{-1}(R_k^{\mu_k}) \text{ for } z \in D(b_k, M_k).$$

Then f has a limit, and hence, a limit 0 at 0 along C_φ, if

$$\lim_{k\to\infty} R_k^{\mu_k} = 0.$$

As in the analytic case, we conclude that if

$$\left(\frac{1-e^{-M_k}}{1+e^{-M_k}}\right)^{\mu_k} \to 0 \text{ as } k \to \infty,$$

then f has the angular limit 0 at 0. ☐

4.8 Remark. Sequential limits for various classes of functions have been extensively studied in the literature. Related results for meromorphic functions are given in [2], for analytic and subharmonic functions in [7], and for quasiregular mappings in [11, 15]. A more general class of Harnack functions has been studied in [14].

References

[1] S. Axler, P. Bourdon, and W. Ramey: *Harmonic Function Theory*, Graduate Texts in Mathematics, 2nd ed., Springer, 2001.

[2] F. Bagemihl and W. Seidel: *Sequential and continuous limits of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. A I 280 (1960), 1–17.

[3] A.F. Beardon: *The geometry of discrete groups*, Graduate Texts in Mathematics 91, Springer-Verlag, New York, 1983.

[4] L. Bieberbach: *Lehrbuch der Funktionentheorie, Bd. 2, Moderne Funktionentheorie*, Leipzig, Teubner, 1931.

[5] P. Duren: *Theory of H^p Spaces*, Academic Press, San Diego, 1970; Dover Publications, Minneola, New York, 2000.

[6] P. Duren: *Harmonic Mappings in the Plane*, Cambridge University Press, 2004.

[7] V. Eiderman and M. Essén: *Uniqueness theorems of analytic and subharmonic functions*, Algebra i Analiz 14, no. 6 (2002), 1–88.

[8] T.W. Gamelin: *Complex Analysis*, New York, Springer-Verlag, 2001.

[9] A. Lyzzaik: *Local properties of some light harmonic mappings*, Canad. J. Math. 44 (1992), 135–153.

[10] M. Mateljević and M. Vuorinen: Private communication.

[11] A. Rasila: *Multiplicity and boundary behavior of quasiregular maps*, Math. Z. 250 (2005), 611–640.

[12] W. Rudin: *Real and complex analysis*, New York, McGraw-Hill, 1987.

[13] D.C. Rung: *Behavior of holomorphic functions in the unit disk on arcs of positive diameter*, J. Math. Kyoto Univ. 8-3 (1968), 417–464.

[14] M. Vuorinen: *On the Harnack constant and the boundary behavior of Harnack functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 259–277.

[15] M. Vuorinen: *Koebe arcs and quasiregular mappings*, Math. Z. 190 (1985), 95–106.

(continued from the back cover)

A570 Janos Karatson, Sergey Korotov
Discrete maximum principles for FEM solutions of nonlinear elliptic systems
May 2009

A569 Antti Hannukainen, Mika Juntunen, Rolf Stenberg
Computations with finite element methods for the Brinkman problem
April 2009

A568 Olavi Nevanlinna
Computing the spectrum and representing the resolvent
April 2009

A567 Antti Hannukainen, Sergey Korotov, Michal Krizek
On a bisection algorithm that produces conforming locally refined simplicial meshes
April 2009

A566 Mika Juntunen, Rolf Stenberg
A residual based a posteriori estimator for the reaction–diffusion problem
February 2009

A565 Ehsan Azmoodeh, Yulia Mishura, Esko Valkeila
On hedging European options in geometric fractional Brownian motion market model
February 2009

A564 Antti H. Niemi
Best bilinear shell element: flat, twisted or curved?
February 2009

A563 Dmitri Kuzmin, Sergey Korotov
Goal-oriented a posteriori error estimates for transport problems
February 2009

A562 Antti H. Niemi
A bilinear shell element based on a refined shallow shell model
December 2008

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The reports are available at *http://math.tkk.fi/reports/* .
The list of reports is continued inside the back cover.

A575 Harri Hakula, Antti Rasila, Matti Vuorinen
On moduli of rings and quadrilaterals: algorithms and experiments
August 2009

A574 Lasse Leskelä, Philippe Robert, Florian Simatos
Stability properties of linear file-sharing networks
July 2009

A573 Mika Juntunen
Finite element methods for parameter dependent problems
June 2009

A572 Bogdan Bojarski
Differentiation of measurable functions and Whitney–Luzin type structure theorems
June 2009

A571 Lasse Leskelä
Computational methods for stochastic relations and Markovian couplings
June 2009

ISBN 978-952-248-043-9 (print)
ISBN 978-952-248-044-6 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)