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Regularity and separation from potential barriers for a non-local phase-field system

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Abstract. We show that solutions of a two-phase model involving a non-local interactive term become more regular immediately after the moment they separate from the pure phases. This result allows us to prove stronger convergence to equilibria. A new proof of the separation property is also given.

Keywords: Non-local phase-field systems; regularity of solutions; separation property; convergence to equilibria.

1 Introduction

This paper is devoted to the study of regularity, separation from singularities, and convergence to equilibria of a two-phase model involving terms non-local in space. Considering a binary alloy with components A and B occupying a spatial domain Ω, and denoting by u and $1-u$ the local concentrations of A and B respectively, the usual model describing the dynamics of phase separation is the Cahn-Hilliard equation. This equation is derived from the free energy functional of the form

$$\mathcal{F}(u) = \int_\Omega \left[f(u) + ku(1-u) + \frac{1}{2}|\nabla u|^2 \right] \, \mathrm{d}x$$

The last term accounts for the interfacial energy. In some situations, it is more adequate to choose an expression, where also the long range interactions are described. This phenomenon is represented by spatial convolution with a suitable kernel, cf. Chen and Fife [6], Giacomin and Lebowitz [13]. It leads to an alternative energy functional of the form

$$F(u) = \int_\Omega \left[f(u) + \kappa u(1-u) + \frac{1}{2}\int_\Omega K(|x-y|)|u(x)-u(y)|^2 \mathrm{d}y \right] \, \mathrm{d}x. \tag{1.1}$$

As shown in [5], this is a natural generalization of $\mathcal{F}$. Namely, we can change variables in the convolution integral using $\eta = \frac{x-y}{2}$, $\xi = \frac{x+y}{2}$, and then expand $u(x) = u(\xi+\eta)$ and $u(y) = u(\xi - \eta)$ around ξ. Taking the first term in the Taylor series of u in the convolution term in (1.1) then gives the energy functional of the same form as $\mathcal{F}$.

With $\kappa = \kappa(x) = \int_\Omega K(|x-y|)\mathrm{d}y$, we rewrite (1.1) in the form

$$F(u) = \int_\Omega \Big[f(u) + u \int_\Omega K(|x-y|)(1-u(t,y))dy \Big] \,\mathrm{d}x. \tag{1.2}$$

Then the chemical potential, defined as the gradient of the energy functional F is given by

$$v = f'(u) + \int_\Omega K(|x-y|)(1-2u(t,y))dy.$$

The model in question then reads:

$$u_t - \nabla \cdot (\mu \nabla v) = 0 \text{ in } (0,T)\times\Omega, \tag{1.3}$$

$$v = f'(u) + \int_\Omega K(|x-y|)(1-2u(t,y))dy, \quad (t,x) \in (0,T)\times\Omega, \tag{1.4}$$

$$\mu\nu \cdot \nabla v = 0 \text{ in } (0,T)\times\partial\Omega, \tag{1.5}$$

$$u(0,x) = u_0, \quad 0 \le u_0(x) \le 1, \quad 0 < \int_\Omega u_0 \,\mathrm{d}x = u_\alpha < 1. \tag{1.6}$$

Here μ denotes a suitable mobility. A natural choice seems to be

$$\mu = \frac{a}{f''(u)}, \quad a \text{ a positive constant}, \tag{1.7}$$

see, e.g., Elliot and Garcke [10].

In the standard case, f is given by

$$f(u) = u \ln u + (1-u)\ln(1-u). \tag{1.8}$$

This implies

$$f'(u) = \ln\Big(\frac{u}{1-u}\Big), \ \mu = \frac{a}{f''(u)} = au(1-u). \tag{1.9}$$

If we denote

$$w = \int_\Omega K(|x-y|)(1-2u(t,y))dy, \tag{1.10}$$

we get

$$(f')^{-1}(v-w) = \frac{1}{1+\exp(w-v)}, \tag{1.11}$$

which gives the *a priori* estimate

$$u \in [0,1]. \tag{1.12}$$

Gajewski and Zacharias [12] proved global existence and uniqueness of weak solutions emanating from initial functions satisfying (1.6). Following their procedure, the same result can be proved for more general f, namely if

$$f \in C^2(0,1) \text{ strictly convex, } \mathrm{Im}(f')^{-1} = [0,1], \ \ \frac{1}{f''} \text{ strictly concave.} \tag{1.13}$$

Convergence (in $L^2(\Omega)$-norm) of any solution to a single equilibrium with f as in (1.8) was proved in [16], using the generalized form of the Łojasiewicz Theorem. The main problem in verifying the assumptions of the Łojasiewicz Theorem was to show that there exists a time T_0 such that solutions separate from the potential barriers 0 and 1, i.e., that there exists $k \in (0, 1)$ such that

$$0 < k < u(t, x) < 1 - k < 1 \quad \text{for a.a. } x \in \Omega, \quad \text{and } t \geq T_0. \tag{1.14}$$

So, the equation (1.3) is not degenerate after that time. With μ as in (1.9), and u satisfying (1.12), $\mu(0, x)$ can vanish, even on a set of a positive measure, so a degeneracy in (1.3) and (1.6) is not excluded at the outset.

In this paper, we prove the separation property for more general f satisfying (1.13), and show that T_0 may be taken arbitrarily small. The proof here is also significantly simpler than that in [16].

With (1.14) at hand, we prove higher regularity of solutions, and, as a consequence, a stronger convergence to equilibria than in [16]. Even if we can write the equation (1.3), after the separation time, in terms of u only, we get a parabolic equation, but we do not have good boundary conditions, so that the maximal regularity is not applicable. Also, it is possible to write a parabolic differential equation for v, but this equation contains the term $|\nabla v|^2$, which prevents us to gain better regularity if the initial value is not sufficiently smooth. The way to improve the regularity of u is to estimate the norm of u_t and then apply a bootstrap argument.

To show that solutions converge in a better norm than in $L^2(\Omega)$, we realize that the higher regularity yields compactness of trajectories in a smaller space, which together with the convergence proved in [16], gives the desired result.

Convergence of solutions of phase-field systems to equilibria was studied by many authors, we can mention only some of them. The first application of the Łojasiewicz inequality to these systems was by Aizicovici, Feireisl and Issard-Roch [3] in the case of a nonisothermal system of second order. The non-local version of the Allen-Cahn equation was studied in Feireisl, Issard-Roch and Petzeltová, [11], where also the non-smooth version of the Łojasiewicz Theorem was proved. Convergence in the conserved phase-field systems with memory was proved in [2], the situation with dynamic boundary conditions was treated, e.g., by Chill, Fašangová and Prüss, [8].

Phase-field systems with singular potentials and the separation property were analyzed, in the non-degenerate situation, in [14], [15]. In the latter paper, a nonlocal (in space) system with inertial term was studied with the assumption that the initial function is already separated from the pure phases. The same assumption was used in the papers by Rocca and Rossi [18], Cherfils, Gatti and Miranville [7]. Cahn-Hilliard equations with singular potentials were studied, by Dupaix [9], Miranville and Zelik [17], from the attractors point of view. Recently, separation from singularities for fourth order parabolic equations were proved by Schimperna and Zelik [19] for nonlinearities of power type. Convergence to equilibrium for Cahn-Hilliard equation with logarithmic potential was studied by Abels and Wilke [1]. Because of the fourth order equation, they have stronger regularity, which enables to show the separation property from some large time on, due to the separation of the equilibria. This is in contrast to our situation, where we have to show the separation. Only then are we able to prove regularity.

The paper is organized as follows. The main results are stated in Section 2, then the new proof of the separation property is given in Section 3. Finally the regularity of solutions, and the strong convergence are proved in Section 4.

2 Preliminaries and Main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and ν the outer unit normal on $\partial\Omega$. Denote by $H^1(\Omega) = W^{1,2}(\Omega)$, $\langle .,. \rangle$ the pairing between $H^1(\Omega)$ and its dual $H^1(\Omega)^*$. Denote

$$H_0^1(\Omega) = \{f \in H^1(\Omega);\ \int_\Omega f(x)\mathrm{d}x = 0\},$$

and

$$H_0^{-1}(\Omega) = (H_0^1)^*(\Omega).$$

To simplify the notation, we omit Ω in the following text, when no confusion can take place. We equip H_0^1 with the inner product

$$(f,g)_{H_0^1} = (\nabla f, \nabla g)_{L^2}.$$

The Riesz isomorphism is then given by the negative Laplacian with Neumann boundary conditions

$$\langle -\Delta_N f, g\rangle_{(H_0^{-1},H_0^1)} = (f,g)_{H_0^1} = (\nabla f, \nabla g)_{L^2}.$$

It means that the inner product in H_0^{-1} is given by

$$(f,g)_{H_0^{-1}} = (\Delta_N^{-1} f, \Delta_N^{-1} g)_{H_0^1} = (\nabla\Delta_N^{-1} f, \nabla\Delta_N^{-1} g)_{L^2},$$

and

$$\|f\|_{L^2}^2 = -(\nabla\Delta_N^{-1} f, \nabla f)_{L^2} \leq \|f\|_{H_0^{-1}} \|f\|_{H_0^1}.$$

We will also use the following consequence of the Gagliardo-Nirenberg inequality:

$$\|\xi\|_{L^2}^2 \leq \varepsilon\|\nabla\xi\|_{L^2}^2 + C\varepsilon^{-n/2}\|\xi\|_{L^1}^2,\ \varepsilon \in (0, \frac{1}{2}), \tag{2.1}$$

and the following version of the Poincaré inequality:

$$\left\| z - \frac{1}{|\Omega_1|}\int_{\Omega_1} z(x)\ \mathrm{d}x \right\|_{L^2} \leq C\frac{1}{|\Omega_1|}\|\nabla z\|_{L^2}, \quad \Omega_1 \subset \Omega, \tag{2.2}$$

which is a particular case of [20, Lemma 4.3.1].

By C we denote a generic constant, which may vary even within one line.

The existence of global weak solutions of the problem (1.3)-(1.6) satisfying

$$\int_0^\infty \Big[\langle u_t, h\rangle + \int_\Omega \mu\nabla v\cdot\nabla h\ \mathrm{d}x\Big]\mathrm{d}t = 0 \ \text{ for all } h \in L^2((0,\infty); H^1(\Omega)),$$

$$u \in C((0,\infty); L^\infty(\Omega)) \cap L^2((0,T); H^1(\Omega)),\ u_t \in L^2((0,\infty); H^1(\Omega)^*), \tag{2.3}$$

$$w \in C((0,\infty); W^{1,\infty}(\Omega)),\ \|w(t)\|_{W^{1,\infty}(\Omega)} \leq B_w \ \text{ for all } \ t \geq 0, \tag{2.4}$$

$$\int_0^\infty \int_\Omega \mu|\nabla v|^2\ \mathrm{d}x < \infty, \tag{2.5}$$

was proved in [12] with f given by (1.8). The following assumptions were imposed on K.

$$\int_\Omega \int_\Omega |K(|x-y|)| \,\mathrm{d}x\,\mathrm{d}y = k_0 < \infty, \ \sup_{x\in\Omega} \int_\Omega |K(|x-y|)|\mathrm{d}y = k_1 < \infty, \tag{2.6}$$

and the operator $\mathcal{J}$ defined by $\mathcal{J}z = -2\int_\Omega K(|x-y|)z(y)\mathrm{d}y$ satisfies

$$\|\mathcal{J}z\|_{W^{1,p}} \le r_p \|z\|_{L^p(\Omega)}, \quad 1 \le p \le \infty. \tag{2.7}$$

As to the global behavior of solutions, the strong convergence to a single equilibrium in the $L^2(\Omega)$-norm was proved in [16]. The equilibrium is a triple u^*, v^*, w^* satisfying

$$u^*(x) = \frac{1}{1+\exp(w^*(x)-v^*)}, \quad w^*(x) = \int_\Omega K(|x-y|)(1-2u^*(y))\mathrm{d}y, \ v^* = \text{constant}. \tag{2.8}$$

The following Theorem was proved in [16]:

Theorem 2.1 *Let f, μ, K satisfy (1.8), (1.7), (2.6), (2.7). Let the triple (u, v, w) be a solution of the problem (1.3)-(1.6) in the sense of (2.3)-(2.5). Then there exist $T_0 \ge 0, B > 0, k > 0$ such that*

$$k \le u(t,x) \le 1-k \textit{ for a.a. } x \in \Omega, \textit{ and } t \ge T_0, \tag{2.9}$$

$$\|v(t)\|_{L^\infty(\Omega)} \le B \quad \textit{for all } t \ge T_0. \tag{2.10}$$

Moreover, there is (u^*, v^*, w^*) satisfying (2.8) such that

$$u(t) \to u^* \quad \textit{strongly in} \quad L^2(\Omega),$$
$$v(t) \to v^* \quad \textit{strongly in} \quad L^2(\Omega),$$
$$w(t) \to w^* \quad \textit{strongly in} \quad H^1(\Omega),$$

as $t \to \infty$.

We will show that the same result holds also for f satisfying only (1.13), and that T_0 can be taken arbitrarily small. Thus, in what follows, we will assume that f satisfies (1.13), and that

$$\mu(z) = \frac{a}{f''(z)}, \quad \frac{\mu(z)}{z(1-z)} \le B_\mu \quad \text{for} \quad z \in (0,1). \tag{2.11}$$

Without loss of generality, we take $a = 1, \ |\Omega| = 1$.

For the regularity of solutions, we will also need

$$\|\mathcal{J}z\|_{W^{2,2}} \le r_2 \|z\|_{W^{1,2}} \ , \tag{2.12}$$

$$f \in C^3(0,1) \tag{2.13}$$

If (2.12) is satisfied, then

$$u \in L^\infty((0,\infty); W^{1,2}) \Rightarrow \|w\|_{L^\infty((0,\infty);W^{2,2})} \le B_w^1. \tag{2.14}$$

The main results are summed up in the following Theorem. Under the additional assumptions (2.12) and (2.13) we obtain more regularity on u and, as a consequence, a stronger asymptotic result if f is analytic.

Theorem 2.2 *Let f, K, μ satisfy (1.13), (2.13), (2.6), (2.7), (2.12), (2.11). Let the triple (u, v, w) be a solution of the problem (1.3)-(1.6). Let $T_0 > 0$. Then there exists $k > 0$ such that*

$$u \in L^\infty((T_0, \infty); W^{2,2}(\Omega)), \tag{2.15}$$

and (2.9) holds. Moreover, if f is real analytic in $(k-\delta, 1-k+\delta)$ for some $\delta \in (0, k)$, and $t \to \infty$, we get

$$u(t) \to u^* \text{ strongly in } \mathcal{B}, \tag{2.16}$$

for any space $\mathcal{B}$ such that $W^{2,2}(\Omega)$ is compactly embedded in $\mathcal{B}$.

Corollary 2.1 *In the three dimensional case the solution converges in $C(\overline{\Omega})$.*

3 The separation property.

In this section, we show that the solution of our problem separates from the pure phases 0 and 1. This property was in fact obtained in [16], but through an unnecessarily complicated proof. Below we present a significantly simpler proof. Moreover, in [16] the nonlinearity was assumed to be as in (1.8). Here we only assume f to satisfy (1.13), and we show that solutions separate from the pure phases after an arbitrary short time $T > 0$.

In the proof below, we only sketch the parts of the proof common to that of [16], and concentrate on the new, simplified steps.

We prove that, under the assumptions detailed in the previous section, one has that the solution of (1.3)-(1.6) is such that $\ln u(t)$, $\ln(1 - u(t))$, and, consequently, $v(t)$, are bounded in the sense that

$$\ln u,\ \ln(1-u),\ v \in L^\infty([T, \infty); L^\infty(\Omega)). \tag{3.1}$$

The bound is shown to depend on the initial function $u_0(x)$ only through the value of u_α. Thus, e.g., the case $|\{x|\ u_0(x) = 0\}| > 0$ is not excluded.

Proposition 3.1 *Let f, K, μ satisfy (1.13), (2.6), (2.7), (2.11). Let the triple (u, v, w) be a solution of the problem (1.3)-(1.6) satisfying (2.3), (2.4), (2.5). Let $T_0 > 0$ be an arbitrary positive time. Then there exists $k > 0$ depending only on T_0 and u_α, such that*

$$k \le u(t, x) \le 1 - k \text{ for a.a. } x \in \Omega, \text{ and } t \ge T. \tag{3.2}$$

Proof: The way to prove (3.2), is to show first that

$$\|\ln u(t, .)\|_{L^r} < B,\ \|\ln(1 - u(t, .))\|_{L^r} < B \ \text{ for all } t \ge T_0,\ r \in [1, \infty), \tag{3.3}$$

assuming that

$$\ln u_0,\ \ \ln(1 - u_0) \in L^\infty(\Omega). \tag{3.4}$$

The upper bound B in (3.3) is shown to be independent of $t \in [T_0, \infty)$, $r \in [1, \infty)$, and of the pointwise values of $u_0(x)$. The upper bound does depend on T_0 and on u_α - the integral mean of the initial value u_0.

We then approximate u_0 with u_0^n satisfying (3.4) and employ the continuous dependence of solutions ([16, Lemma 2.1]) together with Fatou's Lemma to get

$$\| \ln u(T_0)\|_{L^\infty(\Omega)} \le B, \quad \| \ln(1-u(T_0))\|_{L^\infty(\Omega)} \le B, \tag{3.5}$$

even for u_0 as in (1.6).

So, in what follows, we assume (3.4), prove (3.5) and (3.3).

Assuming (3.4), we get

$$\ln u(t), \quad \ln(1-u(t)) \in L^\infty(\Omega) \ \text{ for all } t \ge 0. \tag{3.6}$$

Indeed, by continuity, there exists a maximal time t_{max} such that (3.6) holds in $[0, t_{max})$. If $t_{max} < \infty$, then our procedure leads to the bound (3.5) with $T_0 = t_{max}$, which contradicts the maximality of t_{max}.

Denote

$$M_r(t) = \| \ln u(t,.)\|_{L^r(\Omega)}, \ t \ge 0, \ r = 1, 2, 3, ... \tag{3.7}$$

We show that $M_r(t) \le B$ for $t \ge T_0$, the proof of $\| \ln(1-u(t,.))\|_{L^r(\Omega)} \le B, \ t \ge T_0$ is analogous.

The proof consists of several steps. First, in Lemma 3.1, we derive a differential inequality for M_r, which yields a bound on the possible growth of $M_r(t)$ with increasing r. Specifically, we show that, for $T \in (0,\infty)$,

$$\sup_{t \ge T} M_r(t) \le B_1(T) r^2, \quad \text{for all } r \in [1,\infty). \tag{3.8}$$

Here $B_1 < \infty$ is independent of r, but does depend on T.

In Lemma 3.2 we prove that, for any $T > 0$, there exists $B_2(T)$ such that

$$M_r(T) \le B_2(T) < \infty, \ \text{for all } r \in [1,\infty). \tag{3.9}$$

Lemma 3.3 is needed to prove Lemma 3.2. Finally, having (3.9), we show that

$$B_2(T) \le B \text{ for all } T \ge T_0, \tag{3.10}$$

with B depending only on T_0.

Lemma 3.1 *Let the assumptions of Proposition 3.1, and (3.4) be satisfied. Then there exists a nonincreasing function $B_1 = B_1(T)$, independent of r, such that (3.8) holds for all $T > 0$. Moreover, there exists $T_1 \ge T_0$ such that $TB_1(T)$ is increasing on (T_1, ∞).*

Proof: We derive a differential inequality for M_r, given by (3.7).

$$\frac{\mathrm{d}}{\mathrm{d}t} M_r(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (-\ln u(t))^r \,\mathrm{d}x\right)^{\frac{1}{r}} = -M_r^{1-r}\int_\Omega \frac{(-\ln u)^{r-1}}{u} u_t \,\mathrm{d}x \tag{3.11}$$

$$= M_r^{1-r}\int_\Omega \nabla\Big(\frac{(-\ln u)^{r-1}}{u}\Big)\mu\nabla v \,\mathrm{d}x = M_r^{1-r}\int_\Omega \nabla\Big(\frac{(-\ln u)^{r-1}}{u}\Big)(\nabla u + \mu\nabla w)\,\mathrm{d}x.$$

To obtain the second line of (3.11), recall that, with the notation in (1.10), (1.7), and taking $a = 1$, we get

$$\mu\nabla v = \nabla u + \mu\nabla w. \tag{3.12}$$

For $r = 1$ we have, by the second part of (2.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(t) = -\int_\Omega |\nabla \ln u(t)|^2 \,\mathrm{d}x - \int_\Omega \frac{\mu}{u}(t)\nabla \ln u(t)\nabla w(t)\,\mathrm{d}x$$
$$\leq -\frac{1}{2}\int_\Omega |\nabla \ln u(t)|^2 \,\mathrm{d}x + C.$$

To estimate $\int_\Omega |\nabla \ln u|^2 \,\mathrm{d}x$, we use the generalized version of the Poincaré inequality (2.2). Denote

$$\Omega_1^t = \{x \in \Omega;\ u(t,x) \geq \frac{1}{2}u_\alpha\}.$$

Then, necessarily,

$$|\Omega_1^t| \geq \frac{1}{2}u_\alpha \quad \text{for all } t \geq 0, \tag{3.13}$$

and we have

$$\int_\Omega |\nabla \ln u(t)|^2 \,\mathrm{d}x \geq C|\Omega_1^t|^2\Big[\frac{1}{2}\Big(\int_\Omega |\ln u(t)| \,\mathrm{d}x\Big)^2 - \Big|\ln\frac{u_\alpha}{2}\Big|^2\Big]. \tag{3.14}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(t) \leq -c_1 M_1^2(t) + c_2,$$

where the constants depend on u_α, the integral mean of u_0, but not on the pointwise size of the initial function. Hence M_1 can be dominated by a solution of the ordinary differential equation of the form (3.19) with $b = 0$, which is bounded for $t \geq T$ independently of the size of the initial value.

$$M_1(t) \leq m_1 \quad \text{for all } t \geq T. \tag{3.15}$$

To show the similar estimate for $r > 1$, we continue in (3.11).

$$\frac{\mathrm{d}}{\mathrm{d}t}M_r = -M_r^{1-r}\int_\Omega \Big[(r-1)(-\ln u)^{r-2} + (-\ln u)^{r-1}\Big]\Big[|\nabla \ln u|^2 + \nabla(\ln u)\frac{\mu}{u}\nabla w\Big]\,\mathrm{d}x$$
$$\leq -M_r^{1-r}\int_\Omega [(r-1)(-\ln u)^{r-2} + (-\ln u)^{r-1}]\frac{1}{2}|\nabla \ln u|^2 \,\mathrm{d}x \tag{3.16}$$
$$+M_r^{1-r}\int_\Omega \Big[(r-1)(-\ln u)^{r-2} + (-\ln u)^{r-1}\Big]\frac{1}{2}B_\mu^2 B_w^2 \,\mathrm{d}x.$$

Taking into account that

$$(-\ln u)^{r-2}|\nabla \ln u|^2 = \frac{4}{r^2}|\nabla(-\ln u)^{\frac{r}{2}}|^2,\ \ (-\ln u)^{r-1}|\nabla \ln u|^2 = \frac{4}{(r+1)^2}|\nabla(-\ln u)^{\frac{r+1}{2}}|^2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_r \leq -M_r^{1-r}\frac{1}{r}\int_\Omega \Big|\nabla(-\ln u)^{\frac{r}{2}}\Big|^2 \,\mathrm{d}x - M_r^{1-r}\frac{2}{(r+1)^2}\int_\Omega \Big|\nabla(-\ln u)^{\frac{r+1}{2}}\Big|^2 \,\mathrm{d}x$$
$$+M_r^{1-r}C_1[M_{r-1}^{r-1} + (r-1)M_{r-2}^{r-2}]. \tag{3.17}$$

Drop the first term on the right side of (3.17). Then apply the inequality (2.1) to estimate the second term. As to the last two terms, note that trivially $M_{r-1} \le M_r$. There follows

$$\frac{\mathrm{d}}{\mathrm{d}t} M_r(t) \le -M_r^{1-r}(t) \frac{2}{\varepsilon (r+1)^2} \int_\Omega (-\ln u(t))^{r+1} \, \mathrm{d}x$$

$$+C_2 M_r^{1-r}(t) \varepsilon^{\frac{-n-2}{2}} \Big(\int_\Omega (-\ln u(t))^{\frac{r+1}{2}} \, \mathrm{d}x \Big)^2 + C_1 [1 + (r-1) M_r^{-1}(t)].$$

Apply the interpolation inequality to estimate $M_{\frac{r+1}{2}}$ above in terms of M_r and M_1. One then has

$$\frac{\mathrm{d}}{\mathrm{d}t} M_r \le -C_3 \frac{1}{r^2} M_r^2 + C_4 M_r M_1 + C_1 r.$$

We conclude that M_r can be dominated by solutions of the ordinary differential equation

$$\dot{d}(t) = -a d^2(t) + b\, d(t) + c,$$

where

$$a = C_3/r^2, \quad b = C_4 m_1, \quad c = C_1 r, \tag{3.18}$$

m_1 given by (3.15). The solution of this equation is given by the formula

$$d(t) = \frac{\beta \kappa \exp(a(\beta - \alpha) t) - \alpha}{\kappa \exp(a(\beta-\alpha)t) - 1} \quad \text{if } \ d(0) > \beta \text{ or } d(0) < \alpha, \tag{3.19}$$

$$d(t) = \frac{\beta \kappa \exp(a(\beta - \alpha) t) + \alpha}{\kappa \exp(a(\beta-\alpha)t) + 1} \quad \text{if } \ d(0) \in (\alpha, \beta),$$

with the equilibria

$$\alpha = \frac{b}{2a} - \Big(\frac{c}{a} + \frac{b^2}{4a^2} \Big)^{\frac{1}{2}}, \quad \beta = \frac{b}{2a} + \Big(\frac{c}{a} + \frac{b^2}{4a^2} \Big)^{\frac{1}{2}}.$$

The constant κ is such that the initial condition at $t = 0$ is satisfied. It tends to 1 if $d(0) \to +\infty$. As the function d is decreasing for $d(0) > \beta$, and bounded for $0 \le d(0) \le \beta$ this gives us the estimate

$$d(t) \le \frac{\beta \exp(a(\beta-\alpha)T) - \alpha}{\exp(a(\beta-\alpha)T) - 1} \quad \text{for all } t \ge T,$$

regardless of the initial value $d(0)$. With a, b, c as in (3.18) we arrive at (3.8) with B_1 decreasing on $(0, \infty)$, and $T B_1(T)$ increasing for T large enough.

q.e.d.

The final step is to show (3.3). We begin with the following

Lemma 3.2 *Let the assumptions of Lemma 3.1 be satisfied, and let $T > 0$. Then there exists $B_2 = B_2(T) > 0$ such that*

$$M_r(T) \le B_2(T), \quad \text{for all } r \in [1, \infty), \tag{3.20}$$

where B_2 depends only on T, u_α.

Proof: We show that $M_{2^k},\ k=0,1,2,...$ can be dominated by y_k on a suitably chosen time interval $(t_k,T]$, $t_k>0$, where

$$y_k(t)=\frac{1}{t-t_k}, \tag{3.21}$$

and $\{t_k\}_{k=0}^{\infty}$ an increasing sequence such that $t_k\nearrow t_\infty<T$. Having this, we get

$$M_{2^k}(T)\le y_k(T)\le\frac{1}{T-t_\infty},\ \ k=1,2,3,..., \tag{3.22}$$

and, consequently, (3.20) for all r by interpolation. The functions y_k satisfy

$$y_k'(t)=-y_k^2(t),\ \ y_k(T)=\frac{1}{T-t_k},$$

$$y_k(t)\to\infty \text{ for } t\searrow t_k,\ \ y_k(t)\ge c_k y_{k-1}(t),\ t_k<t\le T,\ \ c_k=\frac{T-t_{k-1}}{T-t_k}. \tag{3.23}$$

Define $t_0<T,\ c_0>1$ such that (B_1 as in Lemma 3.1)

$$T>[c_0B_1(T)]^{-1},\ t_0=T-[c_0B_1(T)]^{-1},\ \text{ and }\ B_1(t_0)\le c_0B_1(T). \tag{3.24}$$

This is possible because B_1 is nonincreasing with time. We show (by induction), that we can choose t_k such that $t_k\nearrow t_\infty<T$ and such that

$$M_{2^k}(t)\le y_k(t),\ t\in(t_k,T). \tag{3.25}$$

Our choice of t_0,c_0, (see (3.24)), together with (3.8) shows that

$$M_1(t)\le B_1(t_0)\le c_0B_1(T)=(T-t_0)^{-1}\le(t-t_0)^{-1}=y_0(t),\ t_0<t\le T,$$

so (3.25) holds for $k=0$. For the induction step we need the following

Lemma 3.3 *Let the assumptions of Lemma 3.2 be satisfied. Let $t_0>0$ be as in (3.23). Let $\tau>t_0,\ r\ge 2$ be such that $\frac{\mathrm{d}M_r}{\mathrm{d}t}$ exists at $t=\tau$ and satisfies*

$$\frac{\mathrm{d}M_r(\tau)}{\mathrm{d}t}+M_r^2(\tau)\ge 0,\ \ M_r(\tau)\ge\frac{1}{\sqrt{r}}. \tag{3.26}$$

Then there is a constant $Q=Q(t_0)$ independent of r such that

$$M_r(\tau)\le Q^{\frac{1}{r}}r^{\frac{3n}{2r}}M_{\frac{r}{2}}(\tau). \tag{3.27}$$

Proof: Suppose that (3.26) is satisfied. Then, referring to (3.17), we get

$$-M_r^2(\tau)\le\frac{\mathrm{d}}{\mathrm{d}t}M_r(\tau) \tag{3.28}$$

$$\le-\frac{1}{r}M_r^{1-r}(\tau)\cdot\int_\Omega\left|\nabla(-\ln u(\tau))^{\frac{r}{2}}\right|^2\,\mathrm{d}x+C_1((r-1)M_r^{-1}(\tau)+1).$$

We multiply both sides by $-rM_r^{r-1}$, and use (3.8) to get

$$\int_\Omega\left|\nabla(-\ln u(\tau))|^{\frac{r}{2}}\right|^2\,\mathrm{d}x\le rM_r^{r+1}(\tau)+C_1r\Big(rM_r^{r-2}(\tau)+M_r^{r-1}(\tau)\Big)\le B_5r^3M_r^r(\tau), \tag{3.29}$$

where B_5 (essentially equal to $B_1(t_0)$) is independent of r. Now apply the estimate (2.1) with $\xi = |\ln u(\tau)|^{\frac{r}{2}}$ and $\epsilon = (2B_5 r^3)^{-1}$, to obtain a lower bound for the left side of (3.29). Combining this with (3.29) and simplifying yields (3.27).

q.e.d.

We proceed to prove Lemma 3.2. Take $r = 2^k,\ k$ a positive integer, and let

$$c_k \overset{\text{def}}{=} (Q(t_0))^{\frac{1}{2^k}} (2^k)^{\frac{3n}{2^{k+1}}}, \quad t_k = T - \left(B_1(T) \prod_{i=0}^{k} c_i \right)^{-1}.$$

With these values of t_k, define y_k as in (3.21). Note that by (3.24), $t_0 > 0$, and that by this choice of $c_k,\ t_k$ one has that (3.23) is satisfied. Without loss of generality, assume $c_k > 1$, for all k. Straightforward calculations show that $\gamma \overset{\text{def}}{=} \Pi_{k=0}^{\infty} c_k < \infty$, so $y_k(T) \leq (T - t_\infty)^{-1}$ holds for all k with $t_\infty = \gamma^{-1}$.

Assume that (3.25) is true for $k-1$. We show, by contradiction, that it holds for k.

Let $M_{2^k}(\tilde{t}) > y_k(\tilde{t})$ for some $\tilde{t} \in (t_k, T)$. Then there is $\hat{t} \in (t_k, \tilde{t}]$ such that $M_{2^k}(\hat{t}) > y_k(\hat{t})$, $M'_{2^k}(\hat{t})$ exists, and, since y_k blows up as $t \searrow t_k$,

$$M'_{2^k}(\hat{t}) \geq y'_k(\hat{t}) = -y_k^2(\hat{t}) \geq -M_{2^k}^2(\hat{t}).$$

But then, by (3.27), and the choice of c_k, and by (3.23),

$$M_{2^k}(\hat{t}) \leq c_k M_{2^{k-1}}(\hat{t}) \leq c_k y_{k-1}(\hat{t}) \leq y_k(\hat{t}),$$

a contradiction, which proves (3.22), and, consequently, Lemma 3.2.

q.e.d.

To complete the proof of Proposition 3.1, we prove (3.10). We realize that T is arbitrary, the bound of M_r is given by $(T - t_\infty)^{-1}$, which depends only on the choice of c_0, and, by (3.24), on the product $TB_1(T)$. This is eventually an increasing function, which can be observed from the decay rate of B_1 estimated in (3.19). So we can choose c_0 fixed, and then (3.22) holds for any $T > T_0$ with the same t_∞. By interpolation, we get (3.9) with B independent of T for any r, which yields (3.10), and, consequently, (3.3) and (3.1). This finally implies (3.2).

q.e.d.

Remark: The crucial point of the proof was to show (3.5), i.e., the existence of one point, where the solution is separated from the potential barriers. Once (3.5) is obtained, there is an alternative way to show (3.10), namely to apply the Alikakos procedure [4] as in [16] to get M_{2^k} bounded on the whole line. This also yields (3.10), but the procedure is more lengthy.

4 Regularity

In this section, we show that solutions of our problem are more regular after the separation time T_0, and prove Theorem 2.2. In order to derive higher regularity of solutions, we first show that

$$u_t \in L^\infty((t_0, \infty); H_0^{-1}(\Omega)) \cap L^2((t_0, \infty); L^2(\Omega)) \qquad (4.1)$$

for some $t_0 \geq 0$, where t_0 is such that the norm $\|u_t(t_0)\|_{H_0^{-1}}$ is finite.

We will proceed formally; the proof can be made exact by approximation of the $t-$derivative by the corresponding quotient.

We differentiate the equation (1.3) with respect to t, and take the scalar product with $\Delta_N^{-1}u_t$. Recall that u_t has zero mean. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2_{H_0^{-1}} = 2(u_{tt}, u_t)_{H_0^{-1}} = 2(\nabla\Delta_N^{-1}u_{tt}, \nabla\Delta_N^{-1}u_t)_{L^2} = -2(u_{tt}, \Delta_N^{-1}u_t)_{L^2},$$

$$(\nabla((\mu\nabla v)_t), \Delta_N^{-1}u_t)_{L^2} = -((\mu\nabla v)_t, \nabla\Delta_N^{-1}u_t)_{L^2} = -(\mu_t\nabla v + \mu\nabla v_t, \nabla\Delta_N^{-1}u_t)_{L^2}$$

$$= -(\mu_t\nabla w + \mu\nabla w_t, \nabla\Delta_N^{-1}u_t)_{L^2} - (\nabla u_t, \nabla\Delta_N^{-1}u_t)_{L^2}$$

$$= -(\mu_t\nabla w + \mu\nabla w_t, \nabla\Delta_N^{-1}u_t)_{L^2} + \|u_t\|^2_{L^2}.$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2_{H_0^{-1}} + \|u_t\|^2_{L^2} = (\mu_t\nabla w + \mu\nabla w_t, \nabla\Delta_N^{-1}u_t)_{L^2}$$

$$\leq C\|u_t\|_{L^2}\|\nabla\Delta_N^{-1}u_t\|_{L^2} \leq \frac{1}{2}\|u_t\|^2_{L^2} + \frac{C^2}{2}\|u_t\|^2_{H_0^{-1}}.$$

Integrating with respect to t, we get

$$\|u_t(t)\|^2_{H_0^{-1}} + \int_{t_0}^{t}\|u_t(s)\|^2_{L^2}\mathrm{d}s \leq \|u_t(t_0)\|^2_{H_0^{-1}} + C^2\int_{t_0}^{t}\|u_t(s)\|^2_{H_0^{-1}}\mathrm{d}s.$$

By the last part of (2.3), this yields (4.1).

With (4.1) at hand, we can improve the regularity of u_t. We proceed as above, but after differentiating (1.3) with respect to t, we multiply it by u_t instead of $\Delta_N^{-1}u_t$. After integration by parts and taking the boundary conditions (1.5) into account, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2 = (\nabla(\mu\nabla v)_t, u_t) = -(\mu_t\nabla w + \mu\nabla w_t + \nabla u_t, \nabla u_t)_{L^2}$$

$$\leq C\|u_t\|_{L^2}\|\nabla u_t\|_{L^2} - \|\nabla u_t\|^2_{L^2} \leq C\|u_t\|^2_{L^2} - \frac{1}{2}\|\nabla u_t\|^2_{L^2}.$$

Integrating with respect to t, we get

$$\|u_t(t)\|^2_{L^2} + \int_{s}^{t}\|\nabla u_t(\tau)\|^2_{L^2}\mathrm{d}\tau \leq \|u_t(s)\|^2_{L^2} + 2C\int_{s}^{t}\|u_t(\tau)\|^2_{L^2}\mathrm{d}\tau \quad \text{for some } s \geq t_0.$$

and, as $u_t \in L^2((t_0,\infty); L^2(\Omega))$, we deduce that

$$u_t \in L^\infty((s,\infty); L^2) \cap L^2((s,\infty); W^{1,2}). \tag{4.2}$$

Taking $t_0 < T_0$, we get (4.2) with $s = T_0$.

The next step is to show that also

$$\nabla u \in L^\infty((T_0,\infty); L^2). \tag{4.3}$$

To this end, we write

$$\nabla u = \mu \nabla v - \mu \nabla w.$$

The last term belongs to the desired space by (2.4) and (2.11). For $\mu \nabla v$ we have

$$\mu \nabla v \in L^2((T_0, \infty); L^2),$$

by (2.5).

We show that also the time derivative of $\mu \nabla v$ belongs to the same space, which implies (4.3).

$$(\mu \nabla v)_t = \mu_t \nabla v + \mu \nabla v_t$$

$$= \mu_t (\frac{1}{\mu} \nabla u + \nabla w) + \mu (-\frac{1}{\mu^2} \mu_t \nabla u + \frac{1}{\mu} \nabla u_t + \nabla w_t)$$

$$= (1 - 2u) u_t \nabla w + \mu \nabla w_t + \nabla u_t.$$

All these terms belong to $L^2((T_0, \infty), L^2)$, and (4.3) follows.

Finally, we go to the equation (1.3), and rewrite it in terms of u only. This is possible because of the separation property of solution, which means that both μ, $\frac{1}{\mu}$ are bounded away from zero.

$$u_t = \nabla(\mu \nabla v) = \nabla\Big(\mu(\frac{1}{\mu} \nabla u + \nabla w)\Big) = \nabla(\nabla u + \mu \nabla w)$$

$$= \Delta u + (1 - 2u) \nabla u \nabla w + \mu \Delta w. \tag{4.4}$$

By (2.7), (2.12), (4.2) and (4.3) we have

$$\Delta u \in L^\infty((T_0, \infty); L^2(\Omega)).$$

So

$$u \in L^\infty((T_0, \infty); W^{2,2}(\Omega)).$$

It follows that the trajectory of u is compact in any space $\mathcal{B} \supset W^{2,2}$ with compact embedding. Then, taking into account the convergence result from Theorem 2.1, we have, for f analytic in $(k - \delta, 1 - k + \delta)$, $\delta \in (0, k)$:

$$u(t) \to u^* \ \text{ in } \mathcal{B} \ \text{ as } t \to \infty.$$

In particular, if $n = 3$, we have

$$u(t) \to u^* \ \text{ in } C(\overline{\Omega}) \ \text{ as } t \to \infty.$$

This concludes the proof of Theorem 2.2. **q.e.d.**

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