Random conformally invariant curves

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Foreword. This minicourse is about conformally invariant random curves in two dimensional domains. The study of these curves is motivated by critical phenomena in statistical physics: it can be generally argued that models of statistical mechanics at their critical points of continuous phase transitions should have scaling limits which exhibit conformal symmetry. The random curves we consider are the natural candidates for scaling limits of interfaces arising in these models — some such interfaces are illustrated in Figures 1, 2, 3. We will encounter different random curves, which are all described and constructed in a rather similar manner. They have become known collectively as Schramm-Loewner evolutions, stochastic Loewner evolutions, or briefly SLEs.

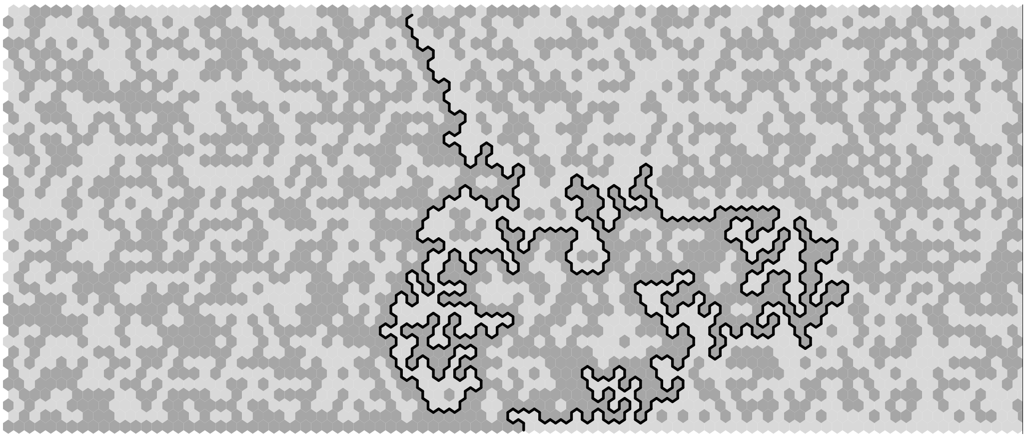


Figure 1: Exploration path of critical percolation on hexagonal lattice separates hexagons of the two different colours.

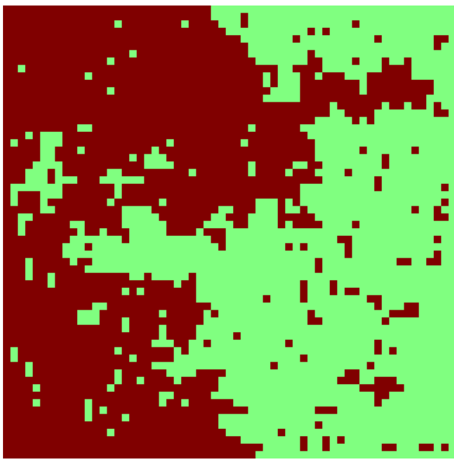


Figure 2: An interface separates different spin clusters in the critical Ising model with Dobrushin boundary conditions (simulation and picture by Antti Kemppainen).

In this short time it is impossible to cover even the crucial parts of the theory in detail, so the aim is rather to introduce the SLE curves and give examples of some of the most common techniques that are needed when working with them. I regret that there will be no time to cover any of the topics in statistical mechanics, which are the real motivation for studying SLEs, and which feature remarkable recent achievements as well as important open research problems. For the reader who is interested in obtaining a more profound understanding of SLEs, there are review articles and overviews [KN04, BB06, Car05], lecture notes [Wer02, Bef10, Law10], a textbook [Law05], and finally of course research articles which would be too numerous to list here. From the ICM talks [Sch06, Smi06, Smi10] one gets a fair picture of the state of the current research.

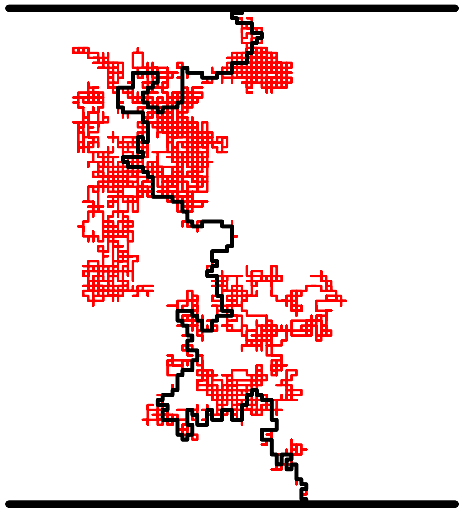


Figure 3: Loop erasure of a random walk.

Acknowledgements. I thank the organizers of the Summer School in Random Geometry, held in Reykjavik at the University of Iceland in August 2011, for the wonderful event and for the invitation to give this minicourse. I very much enjoyed the lectures of the other courses and discussions with other participants — and of course dinners, excursions and beers in good company! I thank those who helped me correct misprints from the original sketch of lecture notes. I thank Antti Kemppainen for letting me use some nice pictures made by him, in particular the one of an interface in the critical Ising model (Figure 2).

Contents

Chapter 1

Tools from complex analysis and stochastic processes

To be able to work with conformally invariant random curves, we need to briefly review some background on conformal geometry and stochastic processes. The goal is merely to mention some of the concepts and results that will be needed later, and to provide references for them.

1.1 On conformal mappings

Below we review some facts about conformal mappings in two dimensions. Most of the statements are proven in complex analysis textbooks such as [Ahl78], and the slightly more advanced topics are treated for example in [Ahl73].

In differential geometry, a mapping between Riemannian manifolds is said to be conformal if the metric becomes multiplied by a positive scalar function. The angles between tangent vectors are independent of such multiplicative factor, and conformal mappings can alternatively be defined as mappings which preserve angles. In two dimensions, for subsets of $\mathbb{R}^2 = \mathbb{C}$, such mappings are well understood — indeed from complex analysis we know that a mapping between subset of $\mathbb{C}$ preserves the magnitude of angles when it is either holomorphic (in which case also the orientation of angles is preserved) or anti-holomorphic (in which case the orientation of angles is reversed). We only consider mappings that preserve the angles with orientation, so for the rest of these notes a conformal mapping from one open set $\Lambda_1 \subset \mathbb{C}$ to another $\Lambda_2 \subset \mathbb{C}$ signifies a bijective holomorphic function $\Lambda_1 \to \Lambda_2$. If there exists a conformal mapping $f : \Lambda_1 \to \Lambda_2$, we call the domains Λ_1 and Λ_2 conformally equivalent. A conformal map $f : \Lambda_1 \to \Lambda_2$ and its inverse $f^{-1} : \Lambda_2 \to \Lambda_1$ are in particular continuous, so conformal equivalence is a stronger notion than homeomorphism.

1.1.1 Simply connected domains and the Riemann mapping theorem

All (non-empty) connected, simply connected open subsets of $\mathbb{C}$ are homeomorphic to each other, and it is a remarkable fact that with the exception of the full complex plane they are all also conformally equivalent. A few examples of such domains, with notation that we will use throughout these notes, are

$$\begin{aligned}
&\text{upper half-plane} && \mathbb{H} = \{z \in \mathbb{C} \;:\; \Im\mathrm{m}(z) > 0\} \\
&\text{unit disk} && \mathbb{D} = \{z \in \mathbb{C} \;:\; |z| < 1\} \\
&\text{horizontal strip} && \mathbb{S} = \{z \in \mathbb{C} \;:\; 0 < \Im\mathrm{m}(z) < \pi\} \\
&\text{rectangle} && (0, L_x) \times (0, L_y) = \Big\{z \in \mathbb{C} \;:\; 0 < \Re\mathrm{e}(z) < L_x,\; 0 < \Im\mathrm{m}(z) < L_y\Big\}
\end{aligned}$$

and so on. We emphasize, however, that the domains need not be as nice as the examples above — domains such as the comb $((0,1)\times(0,1)) \setminus \cup_{n=1}^{\infty}(\mathrm{i}\,\frac{1}{n}, \frac{1}{2} + \mathrm{i}\,\frac{1}{n}])$, or the interior of a fractal Jordan curve exemplify some wilder possibilities. The conformal equivalence of simply connected domains in known as the Riemann mapping theorem, and here is a standard formulation of the statement.

Theorem 1 (Riemann mapping theorem) *Let $\Lambda \subset \mathbb{C}$ be a simply connected open set such that the complement $\mathbb{C} \setminus \Lambda$ is non-empty. Then for any point $z \in \Lambda$ there exists a unique bijective holomorphic function $f : \Lambda \to \mathbb{D}$ such that*

$$f(z) = 0 \qquad \text{and} \qquad f'(z) > 0.$$

Exercise 1 *Show that $\mathbb{C}$ is not conformally equivalent to a simply connected proper subdomain $\Lambda \subsetneq \mathbb{C}$.*

To obtain a conformal map $f : \Lambda_1 \to \Lambda_2$ we may use the above theorem to get conformal maps $f_1 : \Lambda_1 \to \mathbb{D}$ and $f_2 : \Lambda_2 \to \mathbb{D}$, and then set $f = f_2^{-1} \circ f_1$.

The theorem also implies that a conformal map between two simply connected domains is not unique, we were free to choose any z to be mapped to 0, and furthermore we could still rotate the image by multiplying by any complex number of modulus one: the mapping $z \mapsto e^{i\theta} f(z)$ would be conformal $\Lambda \to \mathbb{D}$ and map $z \mapsto 0$, but the derivative at z would be on the half line $e^{i\theta}\, \mathbb{R}_+$.

The non-uniqueness of the conformal map between two simply connected domains corresponds of course to the existence of nontrivial conformal self-maps of any of these domains: if f and $\tilde{f}$ are two different conformal maps $\Lambda \to \mathbb{D}$, then $f^{-1} \circ \tilde{f}$ is a conformal self map of Λ which is not the identity. It is useful to recall the explicit form of conformal self maps of some of the simplest domains.

Lemma 1 *Conformal maps $f : \mathbb{D} \to \mathbb{D}$ are precisely the functions of the form*

$$f(z) = u \, \frac{z - a}{1 - \bar{a} z},$$

where u and a are complex parameters such that $|a| < 1$ and $|u| = 1$.

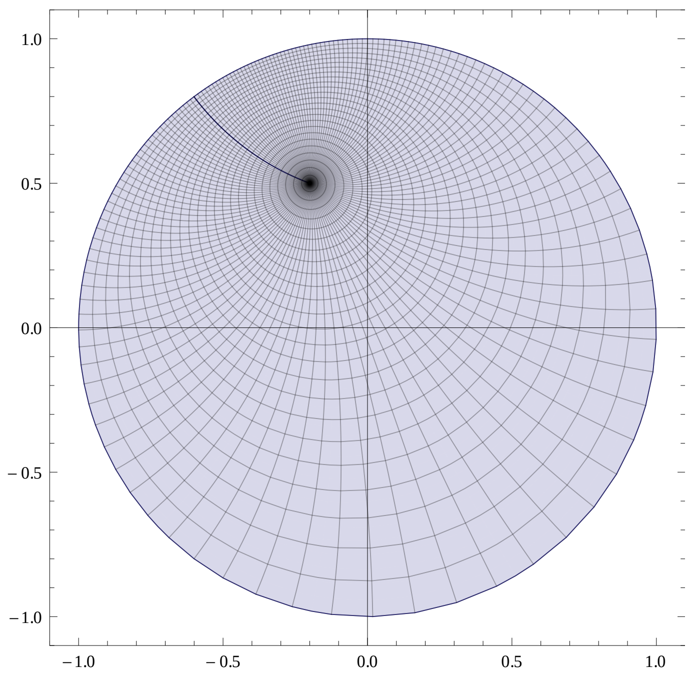


Figure 1.1: The radial coordinate system in $\mathbb{D}$ after the application of a conformal self map of $\mathbb{D}$.

Figure 1.1 illustrates the image of the radial coordinate system of $\mathbb{D}$ by a conformal self map $f : \mathbb{D} \to \mathbb{D}$. The radii and circles intersect at right angles, and by conformality, so do their images.

Lemma 2 *Conformal maps $f : \mathbb{H} \to \mathbb{D}$ are precisely the functions of the form*

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$ are parameters such that $ad - bc > 0$ (it is always possible to normalize $ad - bc = 1$).

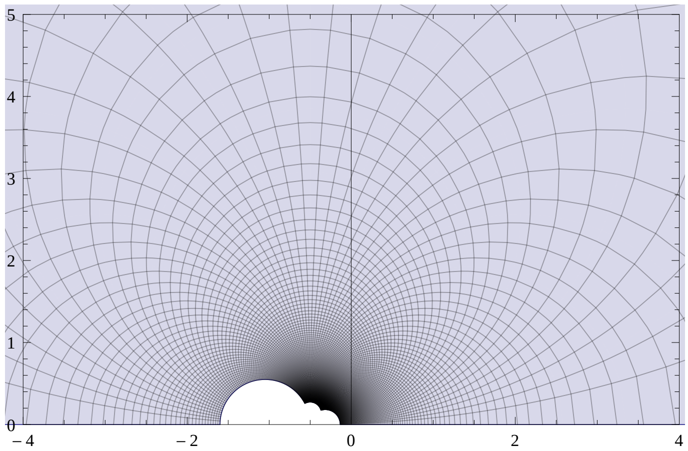


Figure 1.2: A conformal self map of the upper half-plane.

Figure 1.2 illustrates the image of the Euclidean coordinate system of $\mathbb{H}$ by a conformal self map $f : \mathbb{H} \to \mathbb{H}$. The horizontal and vertical lines intersect at right angles, and again by conformality, so do their images.

Let us still give two concrete examples of conformal maps between two different simply connected domains.

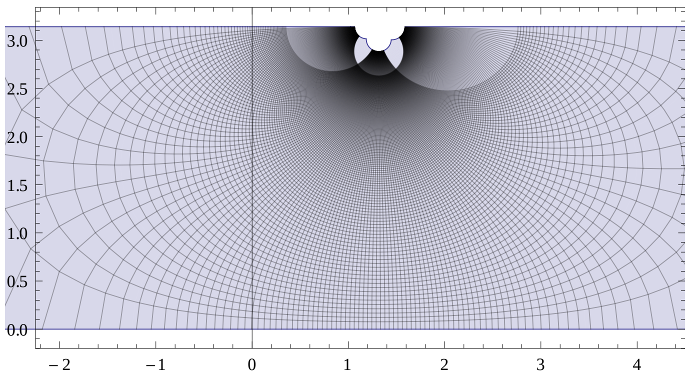


Figure 1.3: Logarithm composed with Möbius transformation maps $\mathbb{H}$ to $\mathbb{S}$.

First, the map $z \mapsto \exp(z)$ maps the horizontal strip $\mathbb{S} = \{x + \mathrm{i}\, y \;:\; x \in \mathbb{R},\; 0 < y < \pi\}$ to the upper half-plane $\mathbb{H}$, and it is clearly holomorphic and bijective. In the other direction, if log denotes the branch of logarithm corresponding to the choice $\arg \in [0, 2\pi)$, then $z \mapsto \log(z) = \log|z| + \mathrm{i}\, \arg(z)$ is conformal $\mathbb{H} \to \mathbb{S}$. Any conformal map from $\mathbb{H}$ to $\mathbb{S}$ is of the form $\log \circ \mu$, where $\mu : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation, one such map is illustrated in Figure 1.3. The images of horizontal and vertical lines intersect forming right angles as they should.

A conformal map from the rectangle to the half-plane is given by Jacobi's elliptic sine. More precisely, denoting the elliptic modulus by $k \in (0, 1)$, the rectangle is $(-K, K) \times (0, K')$, with $K = \int_0^{\pi/2} (1 - k^2 \sin^2(\theta))^{-1/2}\, \mathrm{d}\theta$ and $K' = \int_0^{\pi/2} (1 - (1-k)^2 \sin^2(\theta))^{-1/2}\, \mathrm{d}\theta$ complete elliptic integrals. The mapping $z \mapsto \mathrm{sn}(z; k)$ then takes this rectangle conformally to $\mathbb{H}$, mapping the corners $-K + \mathrm{i}K'$, $-K$, K, $K + \mathrm{i}K'$ to the points $-k^{-1/2}$, -1, 1, $k^{-1/2}$, respectively. Figure 1.4 illustrates this map.

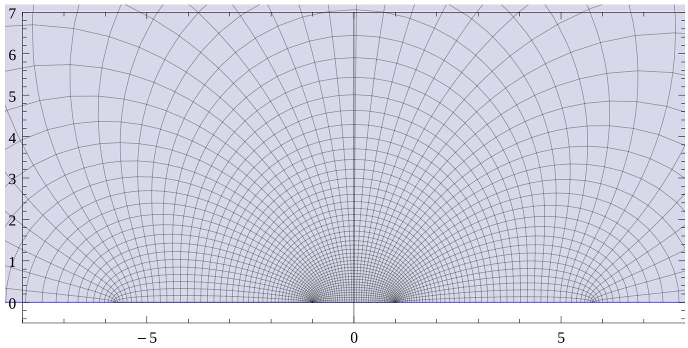


Figure 1.4: Jacobi's elliptic function sn maps a rectangle conformally to the upper half-plane.

Möbius transformations

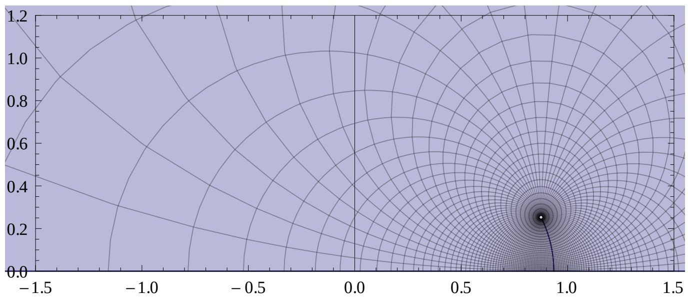


Figure 1.5: Möbius transformations map circles and straight lines to circles or straight lines. This map is from the unit disk to the upper half-plane.

Both of the above Lemmas concerning self maps of a simply connected domain are special cases of the fact that any conformal map between two disks of the Riemann sphere (ordinary disks, half-planes, exteriors of ordinary circles) is a Möbius transformation

$$\mu(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}.$$

Conversely, the image of any circle or straight line under the a nondegenerate ($ad - bc \neq 0$) transformation of the above form is a circle or straight line. As an example, Figure 1.5 features a conformal map $\mathbb{D} \to \mathbb{H}$ and shows the image of the radial coordinate system under this map.

Three real degrees of freedom in choosing a map between simply connected domains

From either of the Lemmas or the Theorem it is clear that choosing a unique conformal map between two simply connected domains amounts to fixing three real parameters (the group of self maps of a simply connected domain is a three dimensional Lie group). We will frequently use for example the following choices:

- If Λ_1, Λ_2 are simply connected domains, $z_1 \in \Lambda_1$ and $z_2 \in \Lambda_2$, and $\theta \in \mathbb{R}$ then there exists a unique conformal map $f : \Lambda_1 \to \Lambda_2$ such that $f(z_1) = z_2$ and $f'(z_1)/e^{i\theta} > 0$. (This follows directly from the statement of Theorem 1.)
- If Λ_1, Λ_2 are simply connected domains, $a_1, b_1, c_1 \in \partial\Lambda_1$ and $a_2, b_2, c_2 \in \partial\Lambda_2$ such that a_j, b_j, c_j appear counter-clockwise along the boundaries of the respective domains Λ_j, then there exists a unique conformal map $f : \Lambda_1 \to \Lambda_2$ such that $f(a_1) = a_2$, $f(b_1) = b_2$, $f(c_1) = c_2$. (An easy way to verify this is to construct a self map of the half plane which maps three boundary points to any desired images.)

* Strictly speaking, for domains with non-Jordan boundary, boundary point is not the appropriate notion, but instead one should map the domain to $\mathbb{D}$ (or some other domain with nice boundary) and consider the preimages of points of $\partial\mathbb{D}$. We leave it for the careful reader to replace the term boundary point by prime end throughtout these notes.

A few convenient ways of fixing the three parameters of conformal maps

We will frequently use the following choices.

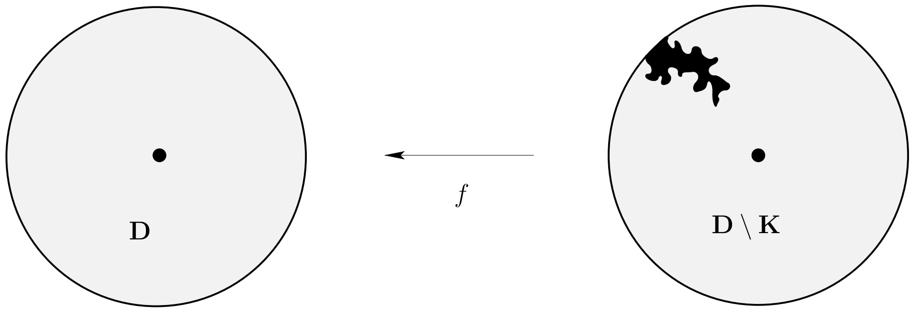


Figure 1.6: A hull in the unit disk, and the map f from its complement back to the disk.

Let us first consider subsets of the unit disk $\mathbb{D}$. If $K \subset \overline{\mathbb{D}}$ is a compact set such that $D = \mathbb{D} \setminus K$ is simply connected, $K = \overline{K \cap \mathbb{D}}$ and $0 \notin K$, then we call K a hull in the unit disk. Figure 1.6 illustrates such a hull. As stated in Theorem 1, we can choose a conformal map whose three degrees of freedom are fixed by requiring the origin to be mapped to the origin and direction of derivative to be positive,

$$f_D : D \to \mathbb{D} \qquad \text{such that} \qquad f_D(0) = 0 \text{ and } f_D'(0) > 0.$$

In fact, applying the Schwarz lemma to $f^{-1} : \mathbb{D} \to D$, we see that $|(f_D^{-1})'(0)| \leq 1$ and consequently $f_D'(0) \geq 1$. The inequality is strict if $K \cap \mathbb{D} \neq \emptyset$. We call the logarithm of the derivative of f_D at the origin the disk capacity of the hull K, and denote

$$\mathrm{cap}_{\mathbb{D}}(K) \;=\; \log f_D'(0) \;\geq\; 0$$

so that

$$f_D'(z) \;=\; e^{\mathrm{cap}_{\mathbb{D}}(K)}\, z + O(z^2).$$

Note that if D_1 and D_2 are two domains of this type with corresponding hulls K_1 and K_2, then

$$\begin{aligned} f_{D_2} \circ f_{D_1} &: f_{D_1}^{-1}(D_2) \to \mathbb{D} \\ (f_{D_2} \circ f_{D_1})'(0) &= f_{D_1}'(0)\, f_{D_2}'(0) \;=\; e^{\mathrm{cap}_{\mathbb{D}}(K_1) + \mathrm{cap}_{\mathbb{D}}(K_2)}, \end{aligned}$$

so in a sense the disk capacity is additive. This shows in particular that if $K' \subset K$ (and for the corresponding domains $D' \supset D$), then $\mathrm{cap}_{\mathbb{D}}(K') \leq \mathrm{cap}_{\mathbb{D}}(K)$, with an equality only if $K' = K$.

Next, let us consider subsets of the upper half-plane $\mathbb{H}$. If $K \subset \overline{\mathbb{H}}$ is a compact set such that $H = \mathbb{H} \setminus K$ is simply connected and $K = \overline{K \cap \mathbb{H}}$, then we call K a hull in the half-plane. Figure 1.7 illustrates such a hull. We can choose a conformal map $\tilde{g} : H \to \mathbb{H}$ which maps the boundary point ∞ to itself, and two real parameters remain, since $z \mapsto a\,\tilde{g}(z) + b$ has the same property. Schwarz reflection

$$\tilde{g}(\bar{z}) \;=\; \overline{\tilde{g}(z)}$$

can be used to extend $\tilde{g}$ to $\mathbb{C} \setminus (K \cup K^*)$, and we then notice that the map $z \mapsto 1/\tilde{g}(1/z)$ has a removable singularity at the origin, in fact a zero. In other words $\tilde{g}$ has a Laurent expansion

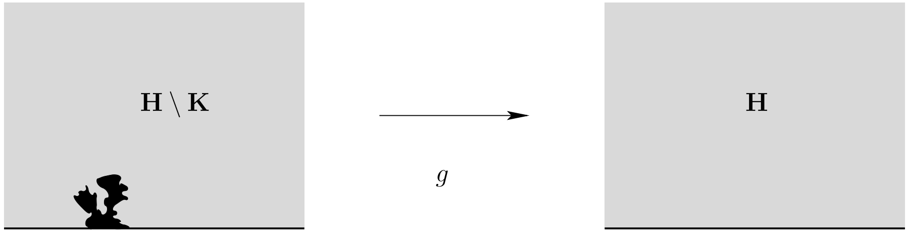


Figure 1.7: A hull in the half-plane, and the map g from its complement back to the half-plane.

$\tilde{g}(z) = cz + d + O(z^{-1})$. The free parameters a, b can be chosen so that $g_H(z) = a\,\tilde{g}(z) + b$ has the expansion

$$g_H(z) \;=\; z + O(z^{-1}),$$

i.e. g_H is as close to identity as possible in neighborhoods of infinity. We call a map with such an expansion hydrodynamically normalized, and we call

$$\mathrm{cap}_{\mathbb{H}}(K) \;=\; \lim_{z\to\infty}\Big(z\,(g_H(z) - z)\Big)$$

the half-plane capacity of the hull K, so that

$$g_H(z) \;=\; z + \frac{\mathrm{cap}_{\mathbb{H}}(K)}{z} + O(z^{-2}).$$

Exercise 2 *Show that* $\mathrm{cap}_{\mathbb{H}}(K) \geq 0$*, with a strict inequality if* $K \cap \mathbb{H} \neq \emptyset$*.*

Note that if H_1 and H_2 are two domains of this type with corresponding hulls K_1 and K_2, then

$$g_{H_2} \circ g_{H_1} \;: g_{H_1}^{-1}(H_2) \to \mathbb{H}$$
$$(g_{H_2} \circ g_{H_1})(z) \;=\; = z + \frac{\mathrm{cap}_{\mathbb{H}}(K_1) + \mathrm{cap}_{\mathbb{H}}(K_2)}{z} + O(z^{-2}),$$

so in a sense the half-plane capacity is additive. In particular, if $K' \subset K$ (and for the corresponding domains $H' \supset H$), then $\mathrm{cap}_{\mathbb{H}}(K') \leq \mathrm{cap}_{\mathbb{H}}(K)$, with an equality only if $K' = K$.

Finally, let us consider subsets of the horizontal strip $\mathbb{S}$. If $K \subset \overline{\mathbb{S}}$ is a compact set such that $S = \mathbb{S} \setminus K$ is simply connected, $K = \overline{K \cap \mathbb{S}}$, then we call K a hull in the strip. We can choose a conformal map $\tilde{h} : S \to \mathbb{S}$ such that the boundary points $\pm\infty$ are preserved, and we still have one real parameter to fix: the maps $z \mapsto \tilde{h}(z) + c$ with $c \in \mathbb{R}$ all preserve the two infinities. One can show that the behavior of the conformal maps at $\pm\infty$ is such that the limits $\lim_{z\to\pm\infty}(\tilde{h}(z) - z)$ exist, so we may choose the translation c such that $h_S(z) = \tilde{h}(z) + c$ satisfies

$$h_S : S \to \mathbb{S} \qquad \text{such that} \qquad \lim_{z\to+\infty}(h_S(z) - z) \;=\; -\lim_{z\to-\infty}(h_S(z) - z).$$

We call such maps strip symmetrically normalized, and we call the quantity

$$\mathrm{cap}_{\mathbb{S}}(K) \;=\; \pm \lim_{z\to\pm\infty}(h_S(z) - z)$$

the strip capacity of the hull K, and one can show that $\mathrm{cap}_{\mathbb{S}}(K) \geq 0$ with strict inequality if $K \cap \mathbb{S} \neq \emptyset$. Note that if S_1 and S_2 are two domains of this type with corresponding hulls K_1 and K_2, then

$$h_{S_2} \circ h_{S_1} \;: h_{S_1}^{-1}(S_2) \to \mathbb{S}$$
$$\lim_{z\to+\infty}(h_{S_2}(h_{S_1}(z)) - z) \;=\; \mathrm{cap}_{\mathbb{S}}(K_1) + \mathrm{cap}_{\mathbb{S}}(K_2),$$

so in a sense the strip capacity is additive. Therefore, in particular, if $K' \subset K$ (and for the corresponding domains $S' \supset S$), then $\mathrm{cap}_{\mathbb{S}}(K') \leq \mathrm{cap}_{\mathbb{D}}(K)$, with an equality only if $K' = K$.

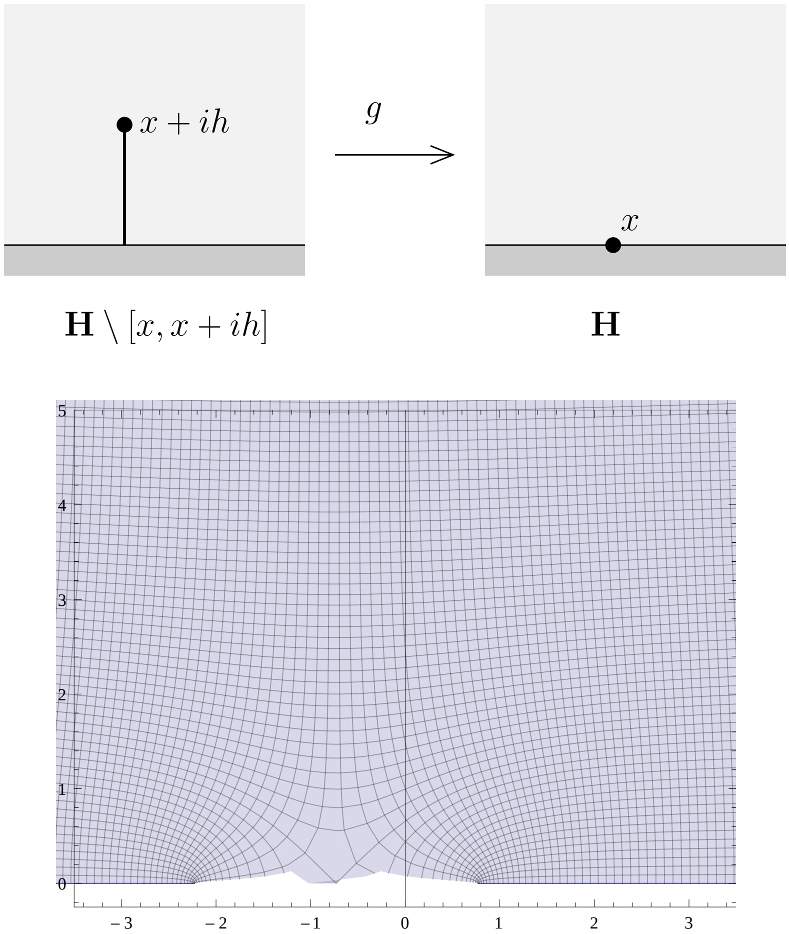


Figure 1.8: The complement $\mathbb{H} \setminus [x, x + \mathrm{i}h]$ of a slit of height h located at x is mapped to the half-plane in such a way that at infinity the map is close to the identity.

A slit map example

Let us give one more example of explicit conformal maps. For $z, w \in \mathbb{C}$, we denote by $[z, w]$ the closed line segment from z to w, i.e. $[z, w] = \{z + s(w - z) : 0 \leq s \leq 1\}$. The complement in the upper half-plane of a vertical segment (slit) starting from the real axis is a simply connected domain $\mathbb{H} \setminus [x, x+\mathrm{i}h]$. Note that the boundary points of the form $x+\mathrm{i}y$, $0 \leq y < h$, can be approached either from the left or the right of the slit, and we interpret these choices as two different boundary points. The reason for this becomes clear when we choose a conformal map to the half-plane. The hydrodynamically normalized conformal map $g : \mathbb{H} \setminus [x, x + \mathrm{i}h] \to \mathbb{H}$ is

$$g(z) \;=\; \sqrt{(z - x)^2 + h^2} + x,$$

where we use the branch of the square root such that $\sqrt{w} \in \mathbb{H}$ for all $w \in \mathbb{C} \setminus [0, \infty)$. A map of this type is illustrated in Figure 1.8. The two ways of approaching the boundary point $x + \mathrm{i}y$, $0 \leq y < h$, have different limits after the conformal map

$$\lim_{\varepsilon \searrow 0} \Big(g(x - \varepsilon + \mathrm{i}\, y) \Big) \;=\; x - \sqrt{h^2 - y^2}$$

$$\lim_{\varepsilon \searrow 0} \Big(g(x + \varepsilon + \mathrm{i}\, y) \Big) \;=\; x + \sqrt{h^2 - y^2},$$

as one easily sees by paying some attention to the branch of the square root.

Our choice of conformal map here is such that far away the map is close to identity, $g(z) = z + O(z^{-1})$. This property manifests itself in Figure 1.8 as the fact that far away the horizontal and vertical lines of the Euclidean coordinate system are only slightly deformed. The two ways of approaching a boundary point on the slit can be obtained by following the (almost) horizontal lines at the same height from far left and from far right: when the height of the lines is less than the height of the slit, these lines end at two (different) points on the interval $[x - h, x + h]$.

1.1.2 Loewner chains

Idea of Loewner chain via infinitesimally changing conformal maps

The Riemann mapping theorem guarantees the existence of conformal maps between any two simply connected domains, but its proof is non constructive, and typically one can only obtain explicit formulas for the conformal maps if the domains are simple enough, as in the examples so far.

Instead, there is a method which describes how the conformal maps vary if the domains are changed by removing a very small piece at a well localized point. Suppose that $\tilde{D} \subset D$ are domains and $D \setminus \tilde{D}$ is a small set located near a point $x \in \partial D$, and let f and $\tilde{f}$ be conformal maps from D and $\tilde{D}$, respectively, to some domain Λ. Since the two domains don't differ by much, we try to choose conformal maps which don't differ by much either. We can write $\tilde{f} = f \circ g$, with $g : \tilde{D} \to D$ is close to the identity

$$g(z) \;\approx\; z + \varepsilon\, v_x(z),$$

where ε measures the size of the small set $D \setminus \tilde{D}$ in an appropriate sense, and $v_x : D \to \mathbb{C}$ is a holomorphic function specifying how we have to move each point to obtain a map $D \to \tilde{D}$. It is more appropriate to think of v_x as a holomorphic vector field

$$v_x(z)\, \partial_z,$$

so that g is the flow of this vector field until time ε determined by the size of the removed piece $D \setminus \tilde{D}$. Note that since ∂D and $\partial \tilde{D}$ coincide except in a small neighborhood of the point x, the flow of the vector field must preserve the boundary, i.e. the vector field must be tangent to the boundary: for $z \in \partial D \setminus \{x\}$

$$v_x(z)\, \partial_z \;\parallel\; \vec{\tau}_z,$$

where $\vec{\tau}_z$ is a tangent vector of ∂D at z. The holomorphic vector field must have some singularity at the boundary point x if the flow of the vector field is to remove the piece $D \setminus \tilde{D}$ located near x. It turns out that the singularity should be a pole, with residue such that $\arg(\mathrm{Res}_{z=x}\, v_x(z)) \;=\; 2\, \arg(\vec{\tau}_x)$. We will outline the argument for this in a concrete case afterwards.

A Loewner chain is a family of such continuously shrinking domains $(D_t)_{t\geq 0}$ and their conformal maps $(g_t)_{t\geq 0}$, $g_t : D_t \to D_0$, for which we can write the infinitesimal change of the conformal maps as a flow of Loewner vector fields: holomorphic vector fields in D_0 which are tangent to the boundary except at the point where the shrinking of the domain is located, where there is a pole.

We will make this idea more precise in concrete cases below, but let us make some comments about the choice of the vector field and give some examples of possible choices. First note that it may be convenient to push forward the vector field from D_0 to some nice reference domain, and do the considerations there — indeed this is necessary already to make sense of the notions of tangent vector and residue on the boundary. Then, regarding the way we prefer to choose the conformal maps g_t: if we want the maps g_t to fix some point $w \in \overline{D}_0$, the vector fields should have a zero at this point. The reader is invited to think of the algebraic restrictions on the number and order of zeros on the boundary or in the interior of the domain. Finally, let us give three nice examples of Loewner vector fields

- In the unit disk $\mathbb{D}$, the vector fields $-z\,\frac{z+x}{z-x}\,\partial_z$ are tangent to the boundary except at $x \in \partial\mathbb{D}$, and they have a simple zero at the interior point 0. These Loewner vector fields are illustrated in Figure 1.9.

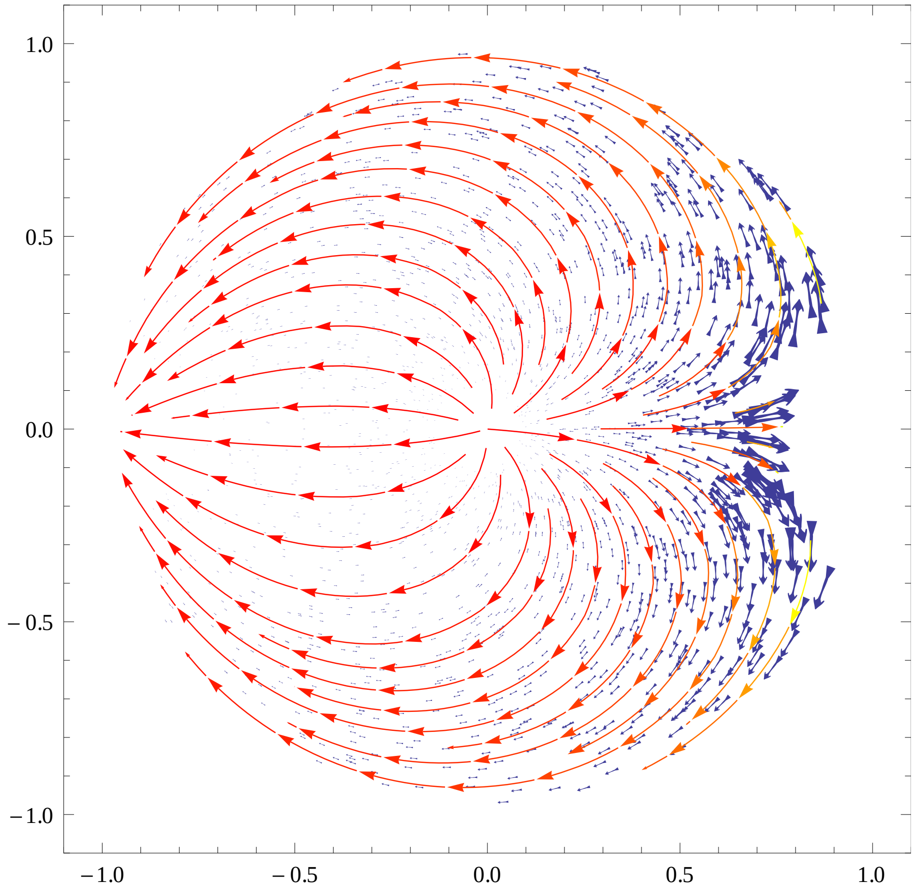


Figure 1.9: The Loewner vector field $-z\,\frac{z+x}{z-x}\,\partial_z$ and its flow in $\mathbb{D}$.

- In the upper half-plane $\mathbb{H}$, the vector fields $\frac{2}{z-x}\,\partial_z$ are tangent to the boundary except at $x \in \mathbb{R}$, and they have a zero of order two at the "boundary point" ∞ (infinity is a boundary point if we view $\mathbb{H}$ as a subset of the Riemann sphere, and to see the order of the zero, one has to choose a local coordinate in a neighborhood of infinity). These Loewner vector fields are illustrated in Figure 1.10.
- In the horizontal strip $\mathbb{S}$, the vector fields $\coth(\frac{z-x}{2})\,\partial_z$ are tangent to the boundary except at $x \in \mathbb{R}$, and they have simple zeros at the "boundary points" $\pm\infty$. These Loewner vector fields are illustrated in Figure 1.11.

We discuss the second case in more detail below.

Hydrodynamically normalized Loewner chain in the half-plane

Let us consider a Loewner chain in the upper half-plane $\mathbb{H}$, and let us choose the conformal maps to be as close to identity at infinity as possible. We consider a family $(H_t)_{t\geq 0}$ of shrinking subdomains:

$$H_0 = \mathbb{H} \qquad \text{and} \qquad s < t \;\Rightarrow H_s \supset H_t.$$

We denote

$$K_t = \overline{\mathbb{H} \setminus H_t},$$

assume that K_t are hulls in the half-plane for all $t \geq 0$. We assume the hulls first of all to be strictly increasing, $K_s \subsetneq K_t$ for $s < t$, and that $K_0 = \emptyset$, and most importantly that the hulls satisfy the following local growth condition

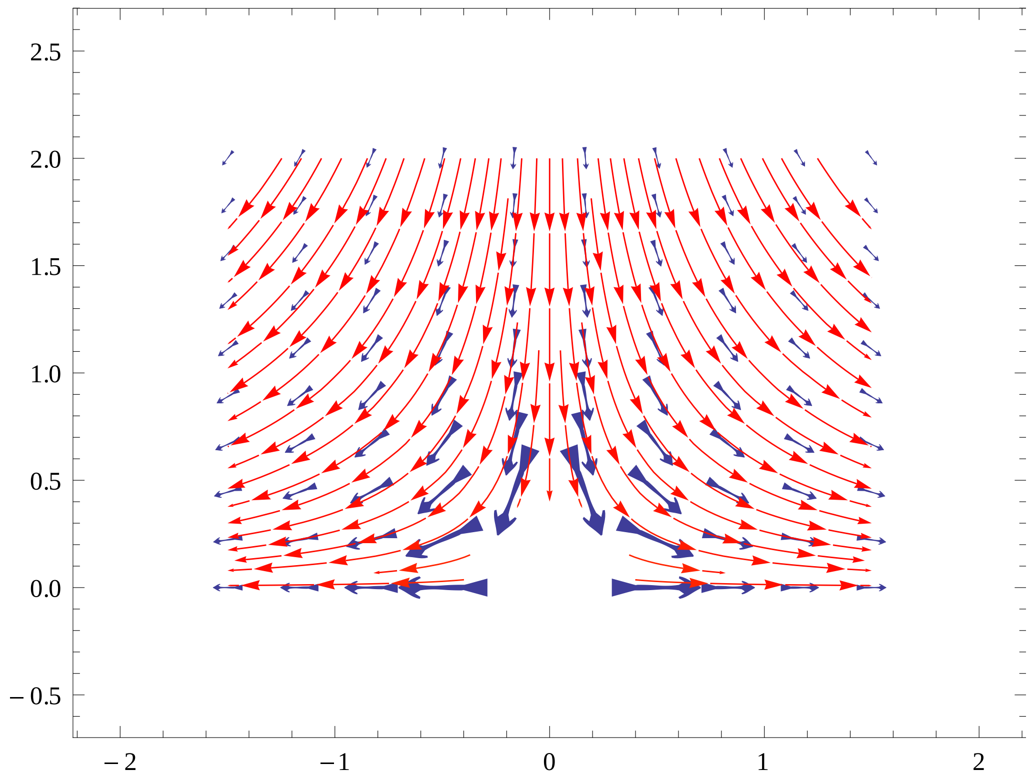


Figure 1.10: The Loewner vector field $\frac{2}{z-x}\,\partial_z$ and its flow in $\mathbb{H}$.

- For all $T > 0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [0, T]$ the piece $K_{t+\delta} \setminus K_t$ can be disconnected from infinity in the domain $H_t = \mathbb{H} \setminus K_t$ by a connected set S of diameter smaller than ε.

We then choose the hydrodynamically normalized conformal maps $g_t : H_t \to \mathbb{H}$,

$$g_t(z) \;=\; z + \frac{\mathrm{cap}_{\mathbb{H}}(K_t)}{z} + O(z^{-2}).$$

The local growth condition can be shown to guarantee that $g_t(z)$ evolves continuously in time t, up to the time when the solution ceases to exist. The half-plane capacities $\mathrm{cap}_{\mathbb{H}}(K_t)$ are therefore continuous and strictly increasing, whence we can assume a time parametrization such that $\mathrm{cap}_{\mathbb{H}}(K_t) = 2t$.

For $0 \leq s$ and $\delta > 0$ the mapping $\tilde{g}_\delta = g_{s+\delta} \circ g_t^{-1} : g_s(H_{s+\delta}) \to \mathbb{H}$ is still hydrodynamically normalized. Consider the harmonic function

$$z \mapsto \Im\mathrm{m}(\tilde{g}_\delta(z) - z) \qquad \mathbb{H} \to \mathbb{R}_+.$$

It's boundary values are 0 except in a neighborhood of the point X_s which is the image of the position of local growth

$$\{X_s\} \;=\; \bigcap_{\delta>0} \overline{g_s(K_{s+\delta} \setminus K_s)}.$$

We can write a representation of the harmonic function using the Poisson kernel $P_{\mathbb{H}}(z;\xi) = -\frac{1}{\pi}\,\Im\mathrm{m}(\frac{1}{z-\xi})$ in the half-plane

$$\Im\mathrm{m}(\tilde{g}_\delta(z) - z) \;\approx\; \text{const.} \times \frac{-1}{\pi}\,\Im\mathrm{m}\Big(\frac{1}{z - X_s}\Big),$$

and consequently

$$\tilde{g}_\delta(z) \;\approx\; z + \frac{\text{const.}}{z - X_s}.$$

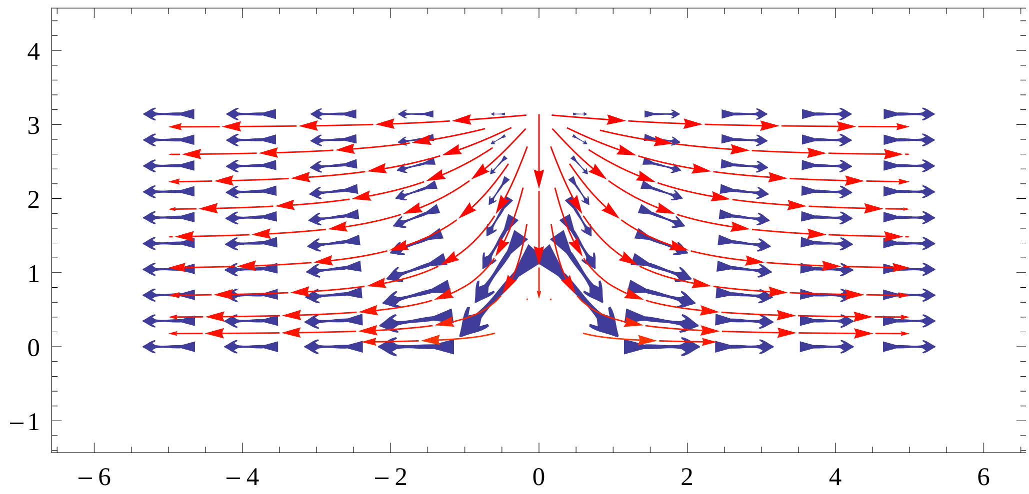


Figure 1.11: The Loewner vector field $\coth(\frac{z-x}{2})\,\partial_z$ and its flow in $\mathbb{S}$.

The constant has to be 2δ, by additivity of the half-plane capacity, so

$$\frac{1}{\delta}\Big(g_{s+\delta}(g_s^{-1}(z)) - g_s(g_s^{-1}(z))\Big) \;\approx\; \frac{2}{z - X_s}.$$

Substituting $z = g_s(w)$ we expect to get

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(w)\Big|_{t=s} \;=\; \frac{2}{g_s(w) - X_s},$$

which indeed could be proved by doing the argument a bit more carefully.

The Loewner chain $(g_t)_{t\geq 0}$ thus satisfies the Loewner differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2}{g_t(z) - X_t}, \tag{1.1}$$

where X_t is the image under g_t of the position of local growth of the hulls, a continuous function which we call the driving function of the Loewner chain (g_t). This is a particular case of flows of Loewner vector fields $v_x(z)\partial_z$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; v_{X_t}(g_t(z)), \tag{1.2}$$

with the vector fields $v_x(z)\partial_z = \frac{2}{z-x}\partial_z$ which are illustrated in Figure 1.10.

Note that the slit map of Figure 1.8 exemplifies this picture: if the driving function is constant $X_t = x$ for all $t \geq 0$, then the solution of $\dot{g}_t(z) = 2/(g_t(z) - x)$ with initial condition $g_0(z) = z$ is clearly $g_t(z) = \sqrt{(z-x)^2 + 4t} + x$, the hydrodynamical conformal map from $\mathbb{H} \setminus [x, x + \mathrm{i}\,2\sqrt{t}]$ to $\mathbb{H}$.

1.2 On stochastic calculus

In these notes we essentially only consider continuous stochastic processes indexed by continuous time t.

1.2.1 Martingales and optional stopping theorem

The information we have about stochastic processes accumulates in time, and this is represented by a filtration $(\mathcal{F}_t)_{t\geq 0}$. The sigma algebra $\mathcal{F}_t$ represents information available at time t, so that in particular for any stochastic process $(X_t)_{t \geq 0}$ the value X_t at time t must be measurable with respect to $\mathcal{F}_t$. The information is accumulating, meaning that sigma algebras become finer, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. Usually we say there's no information available at time zero, so $\mathcal{F}_0$ is the trivial sigma-algebra $\{\emptyset, \Omega\}$. Typically we also consider the information contained in a given process $(X_t)_{t \geq 0}$, in the sense that $\mathcal{F}_t$ is the smallest sigma-algebra with respect to which all $\mathcal{F}_s$, $+ \leq s \leq t$, are measurable.

A stochastic process $(M_t)_{t \geq 0}$ is said to be a martingale (with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$) if the conditional expected value of the future of the process given the information at the present is the same as the present value of the process, i.e. for all $0 \leq s \leq t$

$$\mathsf{E}\Big[M_t \,\Big|\, \mathcal{F}_s\Big] = M_s.$$

Roughly speaking, martingales are stochastic processes which are conserved in mean. In particular, the expected value of a martingale is constant in time: with $s = 0$ the martingale property reads

$$\mathsf{E}\Big[M_t\Big] = M_0 \qquad \text{for any } t \geq 0.$$

A random time τ, whose occurrence before time $t \geq 0$ can be decided with the information available at time t, is called a stopping time. The optional stopping theorem states that if (M_t) is a martingale and τ is a (finite) stopping time, then

$$\mathsf{E}\Big[M_\tau\Big] = M_0.$$

1.2.2 Brownian motion

The standard Brownian motion on $\mathbb{R}$ is the process $(B_t)_{t \geq 0}$ whose finite dimensional marginals are, for $0 < t_1 < t_2 < \cdots < t_n$,

$$\mathsf{P}\Big[B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n\Big] = \int \cdots \int_{A_1 \times \cdots \times A_n} \exp\Big(\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\Big) \frac{1}{\prod_{j=1}^{n} \sqrt{2\pi(t_j - t_{j-1})}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where $t_0 = 0$ and $x_0 = 0$. Figure 1.12 shows a realization of the Brownian motion.

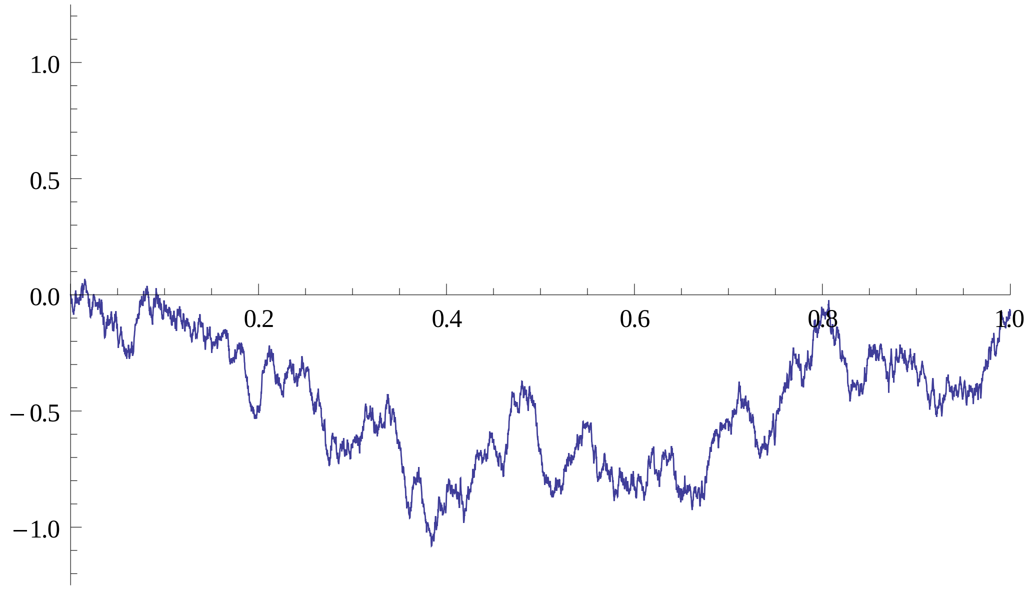


Figure 1.12: Values B_t of a Brownian motion are plotted on the vertical axis against t on the horizontal axis.

A few important properties of the Brownian motion are

- The Brownian motion is the scaling limit of simple random walks: if $S_n = \sum_{j=1}^{n} \xi_j$ with ξ_j i.i.d. $\mathsf{E}[\xi_j] = 0$, $\mathsf{E}[\xi_j^2] = 1$, then the law of the process $(\sqrt{\delta}\, S_{\lfloor t/\delta \rfloor})_{t\geq 0}$ tends to the law of $(B_t)_{t\geq 0}$ as $\delta \searrow 0$.
- For any $\varepsilon > 0$, the function $t \mapsto B_t$ is (almost surely) Hölder continuous with exponent $\frac{1}{2} - \varepsilon$, but it is not Hölder continuous with exponent $\frac{1}{2}$.
- The process $(B_t)_{t\geq 0}$ is centered Gaussian and, the value at time t is mean zero variance t normal distributed, $B_t \sim N(0, \sqrt{t})$.
- For any $s \geq 0$, the increments $B_{s+t} - B_s$ are independent of $B|_{[0,s]}$, and the increment process $(B_{s+t} - B_s)_{t\geq 0}$ has the same law as $(B_t)_{t\geq 0}$.
- The process $(B_t)_{t\geq 0}$ is a martingale.
- The process $(B_t^2 - t)_{t\geq 0}$ is a martingale.

To check the martingale property of $B_t^2 - t$, we write for $0 \leq s < t$

$$B_t^2 \; = \; (B_s + B_t - B_s)^2 \; = \; B_s^2 + 2\, B_s\, (B_t - B_s) + (B_t - B_s)^2$$

and recall that the increment $B_t - B_s$ is independent of $B|_{[0,s]}$. Then compute

$$\begin{aligned}
\mathsf{E}\Big[B_t^2 - t \;\Big|\; B|_{[0,s]}\Big] &= \mathsf{E}\Big[B_s^2 + 2\, B_s\, (B_t - B_s) + (B_t - B_s)^2 - t \;\Big|\; B|_{[0,s]}\Big] \\
&= B_s^2 + 2\; B_s\; \mathsf{E}[(B_t - B_s)] + \mathsf{E}[(B_t - B_s)^2] - t \\
&= B_s^2 + 0 + (t - s) - t \\
&= B_s^2 - s.
\end{aligned}$$

Characterizations of the Brownian motion

The standard Brownian motion on $\mathbb{R}$ could also be defined as the centered Gaussian process $(B_t)_{t\geq 0}$ with covariance

$$\mathsf{E}\Big[B_{t_1}\, B_{t_2}\Big] \; = \; \min\{t_1, t_2\}\,.$$

We will later use the fact that Brownian motion can be characterized using this property of independent stationary increments: if $(X_t)_{t\geq 0}$ is a continuous real valued process with $X_0 = 0$, such that for all $s \geq 0$ the increment process $(X_{s+t} - X_s)_{t\geq 0}$ has the same law as $(X_t)_{t\geq 0}$ and is independent of $X|_{[0,s]}$, then $(X_t)_{t\geq 0}$ has the law of

$$\Big(\sqrt{\kappa}\, B_t + \alpha\, t\Big)_{t\geq 0}$$

for some $\kappa \geq 0$, $\alpha \in \mathbb{R}$.

Another characterization of the Brownian motion by Lévy is the following: if $(X_t)_{t\geq 0}$ and $(X_t^2 - t)_{t\geq 0}$ are continuous martingales and $X_0 = 0$, then X has the law of standard Brownian motion.

1.2.3 Stochastic calculus

We will need to do calculus with differentials of Brownian motion. The reader will find a proper treatment of this stochastic calculus (or Itô's calculus) in any of the textbooks [KS91, Oks02, RY99]. Here we will give an intuitive explanation of the meaning of such calculus and most important formulas for working with it.

We will denote by $\mathrm{d}t$ a differential of the time parameter t of our stochastic processes, intuitively this is to be interpreted as $(\Delta t)_j = t_j - t_{j-1}$, where $0 = t_0 < t_1 < t_2 < \cdots$ is a discretization of time. Expressions involving $\mathrm{d}t$ are always to be integrated (or the increments $(\Delta t)_j$ are to be summed) over time intervals much longer than the mesh $|\Delta| = \max_j (\Delta t)_j$ of the discretization of time. Similarly, we will denote by $\mathrm{d}B_t$ a differential of the Brownian motion, corresponding intuitively to $(\Delta B)_j = B_{t_j} - B_{t_{j-1}}$. Again, expressions involving $\mathrm{d}B_t$ are to be integrated (or the increments $(\Delta B)_j$

are to be summed) over finite time intervals much longer than $|\Delta| = \max_j(\Delta t)_j$. Note that $(\Delta B)_j$ is a centered Gaussian of variance $(\Delta t)_j$, independent of $B|_{[0,t_{j-1}]}$, and it is a good intuition that $\mathrm{d}B_t$ is a centered Gaussian of variance $\mathrm{d}t$ independent of $B|_{[0,t]}$. Thus the of size of $\mathrm{d}B_t$ is of order $\sqrt{\mathrm{d}t}$.

For example, if $\sigma : [0,\infty)\mathbb{R}$ is a given deterministic function, then multiples $\sigma(t_{j-1})\,(\Delta B)_j$ of the small independent centered Gaussians add up to a bigger centered Gaussian, and the expression

$$\int_{s_-}^{s_+} \sigma(t)\,\mathrm{d}B_t$$

is Gaussian with mean 0 and variance $\int_{s_-}^{s_+} \sigma(t)^2\,\mathrm{d}t$, independent of $B|_{[0,s_-]}$. Note however, that in the above example we needed that the coefficients σ don't depend on B.

There is one important thing to pay attention to, the formula $(\mathrm{d}B_t)^2 = \mathrm{d}t$ which may seem counterintuitive — the squared random infinitesimal Brownian increment is deterministic infinitesimal time increment. One naive explanation is as follows. The square of the Brownian increment $((\Delta B)_j)^2$ has the law of $(\Delta t)_j$ times the square of a unit normal random variable. Thus indeed the expected value is the time increment, $\mathsf{E}[((\Delta B)_j)^2] = (\Delta t)_j$. The randomness, on the other hand, has too small scale: the variance of $((\Delta B)_j)^2$ is $2((\Delta t)_j)^2$, so summing over a finite time interval with any bounded coefficients the increments squared results to a random variable which is has variance which tends to zero as the discretization of time gets finer, $|\Delta| = \max_j(\Delta t)_j \searrow 0$, hence $(\mathrm{d}B_t)^2$ becomes deterministic and is simply given by its expected value. Expressions such that $\mathrm{d}B_t\;\mathrm{d}t$ and $(\mathrm{d}t)^2$ are zero, as the corresponding increments are of too small scale.

Itô's formula

We consider stochastic processes $(X_t)_{t\geq 0}$, whose infinitesimal increments have the form

$$\mathrm{d}X_t \;=\; \alpha_t\;\mathrm{d}t + \beta_t\;\mathrm{d}B_t,$$

where (α_t) and (β_t) are some processes (which are predictable with the information about the Brownian motion B up to the corresponding time instant). Such an equation for the increments is called stochastic differential equation, and a more appropriate meaning of it is

$$X_s \;=\; X_0 + \int_0^s \alpha_t\;\mathrm{d}t + \int_0^s \beta_t\;\mathrm{d}B_t,$$

where the integrals in turn are to be understood as limits of discretizations.

Note that the conditional expected value

$$\mathsf{E}\Big[\int_{s_1}^{s_2} \beta_t\;\mathrm{d}B_t\;\Big|\;B|_{[0,s_1]}\Big]$$

is zero, since the integral is a sum of multiples of Brownian increments which are independent of $B|_{[0,s_1]}$ (and the coefficients are independent of the increments due to the predictability requirement). Therefore we expect a process (X_t) with increments $\mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \beta_t\,\mathrm{d}B_t$ to be a martingale if and only if $\alpha \equiv 0$ (this is only slightly too naive because of issues of existence of the expected values).

We often need to know what are the increments of a process which is obtained by applying some function to another process. If $\mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \beta_t\,\mathrm{d}B_t$ and f is a twice continuously differentiable function, then the increments of the process $(f(X_t))_{t\geq 0}$ are given by Itô's formula

$$\mathrm{d}f(X_t) \;=\; f'(X_t)\,\beta_t\;\mathrm{d}B_t + f'(X_t)\,\alpha_t\;\mathrm{d}t + \frac{1}{2}\,f''(X_t)\,\beta_t^2\;\mathrm{d}t.$$

It is easy to understand this formula as a Taylor expansion with the rules $(\mathrm{d}B_t)^2 = \mathrm{d}t$, $(\mathrm{d}t)^2 = 0$ and $\mathrm{d}B_t\;\mathrm{d}t = 0$.

As a first example, we calculate

$$\mathrm{d}(B_t^2 - t) \;=\; 2\,B_t\;\mathrm{d}B_t + \frac{1}{2}\,2\,\mathrm{d}t - \mathrm{d}t \;=\; 2\,B_t\;\mathrm{d}B_t.$$

Indeed, no $\mathrm{d}t$ term remains, in accordance with $B_t^2 - t$ being a martingale.

The Itô's formula easily generalizes to a case where one has several independent Brownian motions and their infinitesimal increments.

Time changes of stochastic processes

1.2.4 Conformal invariance of two-dimensional Brownian motion

As an application of the above techniques, one can prove that the two-dimensional Brownian motion is conformally invariant (up to a time change).

Using the conformal invariance of the two-dimensional Brownian motion and the optional stopping theorem one for example obtains the following probabilistic interpretation of the half plane capacity.

Exercise 3 *Show that if K is a hull in the upper half-plane, $(\mathbf{B}_t)$ is a two-dimensional Brownian motion $\mathbf{B}_t = [B^x_t, B^y_t]^\top$, and $\tau = \inf\{t \geq 0 \;:\; \mathbf{B}_t \in K \cup \mathbb{R}\}$, then*

$$\mathrm{cap}_{\mathbb{H}}(K) \;=\; \mathrm{const.} \times \lim_{y \to \infty} y \, \mathsf{E}_{\mathbf{B}_0 = \mathrm{i}\, y}[\Im\mathrm{m}(\mathbf{B}_\tau)].$$

This formula provides an alternative way of proving positivity of the half-plane capacity.

Chapter 2

Introduction to Schramm-Loewner Evolutions

2.1 Schramm's classification principles

Here we give the argument, due to Oded Schramm [Sch00], ¡¡¡¡¡¡ .mine which identifies SLEs as the appropriate candidates of scaling ======= which identifies SLEs as the correct candidates of scaling ¿¿¿¿¿¿¿ .r1972 limits of interfaces in critical statistical mechanics models. When the setup is such that all the domains we need are conformally equivalent, this Schramm's principle classifies all possible random curves which can be described by Loewner evolutions, which are conformally invariant, and which satisfy a natural Markovian type property. Having found the classification, we then take that as a definition of SLE.

Conformally invariant random curves

Let us now start considering conformally invariant random curves. We thus seek to associate to each domain Λ (with a number of marked points) a probability measure on curves in that domain, such that the push-forward of the probability measure in Λ by a conformal map f from Λ to another domain Λ' coincides with the probability measure associated to Λ'.

In such a consideration, we naturally restrict attention to some class of domains (with marked points) that are conformally equivalent. For concreteness we first discuss one of the simplest conformal types, simply connected domains $\Lambda \subset \mathbb{C}$ with two marked boundary points $a, b \in \partial\Lambda$, and look for (oriented but unparametrized) random curves from a to b in $\overline{\Lambda}$. By the Riemann mapping theorem, if $(\Lambda_1; a_1, b_1)$ and $(\Lambda_2; a_2, b_2)$ are two such domains, there exists a conformal map $f : \Lambda_1 \to \Lambda_2$ such that $f(a_1) = a_2$ and $f(b_1) = b_2$, so this collection of domains indeed forms a conformal equivalence class. Note further that such conformal map is not unique, but there is a one parameter family of conformal self maps of any given domain of this type. The setup of simply connected domains with a random curve connecting two marked boundary points is ofter referred to as the *chordal* case.

In this setup, we seek a collection $(\mathsf{P}_{(\Lambda;a,b)})$ of probability measures associated to domains Λ with marked boundary points $a, b \in \partial\Lambda$, such that for any conformal $f : \Lambda \to f(\Lambda)$ we have

$$f_* \, \mathsf{P}_{(\Lambda;a,b)} \;=\; \mathsf{P}_{(f(\Lambda);f(a),f(b))}.$$

In other words, if a random curve γ (in Λ from a to b) has the law $\mathsf{P}_{(\Lambda;a,b)}$, then its image $f \circ \gamma$ has the law $\mathsf{P}_{(f(\Lambda);f(a),f(b))}$.

Conformal invariance alone is not a very restrictive requirement. Indeed, if we were given *any* probability measure $\mathsf{P}_{\mathrm{ref}}$ on curves in one reference domain $(\Lambda_{\mathrm{ref}}; a_{\mathrm{ref}}, b_{\mathrm{ref}})$, subject only to the condition that $\mathsf{P}_{\mathrm{ref}}$ is invariant under the conformal self maps of the reference domain, then we could *define* the probability measures in $(\Lambda; a, b)$ as $f_* \, \mathsf{P}_{\mathrm{ref}}$, where f is any conformal map from the reference domain to $(\Lambda; a, b)$, and thus we would obtain a conformally invariant collection of probability measures. To find interesting conformally invariant random curves which can also be classified, we impose a further condition of *domain Markov property*, which is motivated by interfaces in statistical mechanics.

The domain Markov property (chordal case)

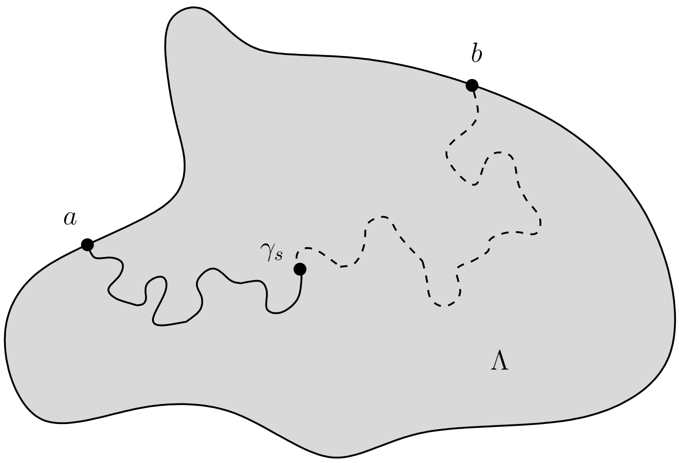


Figure 2.1: The domain Markov property concerns the conditional law of the remaining part of the curve, given an initial segment $\gamma[0,s]$ of it.

We still first consider the chordal setup: random curves from $a \in \partial\Lambda$ to $b \in \partial\Lambda$ in the closure of a simply connected domain Λ. We consider the curves as oriented but unparametrized: two parametrized curves $\gamma_1 : [T_1^-, T_1^+] \to \mathbb{C}$ and $\gamma_2 : [T_2^-, T_2^+] \to \mathbb{C}$ are identified if $\gamma_1 = \gamma_2 \circ \theta$ for some increasing bijection $\theta : [T_1^-, T_1^+] \to [T_2^-, T_2^+]$. An *initial segment* of $\gamma : [T^-, T^+] \to \mathbb{C}$ is a restriction of γ to a subinterval containing the beginning, i.e. $\gamma|_{[T^-,s]}$ with $T^- \leq s \leq T^+$. The *tip* of an initial segment $\gamma|_{[T^-,s]}$ is the point $\gamma(s)$.

The crucial assumption which adds significant content to our considerations is the following

- *Domain Markov property*: We assume that given any initial segment $\gamma|_{[0,s]}$ of the random curve $\gamma : [0,T] \to \overline{\Lambda}$ in $(\Lambda; a, b)$, the conditional law of the remaining part $\gamma|_{[s,T]}$ is the probability measure associated to the domain $(\tilde{\Lambda}; \tilde{a}, b)$, where $\tilde{\Lambda}$ is component containing b of the complement $\Lambda \setminus \gamma[0,s]$ of the initial segment and $\tilde{a} = \gamma(s)$ is the tip of the initial segment. Put in another way,

$$\mathsf{P}_{(\Lambda;a,b)}\Big[\,\cdot\, \Big| \gamma|_{[0,s]} = \eta \Big] \;=\; \eta \boxplus \mathsf{P}_{(\Lambda\setminus\eta[0,s];\eta(s),b)}[\,\cdot\,],$$

  where $\boxplus$ denotes concatenation of curves and $\Lambda \setminus \eta[0,s]$ is understood to stand for the relevant connected component only.

The domain Markov property thus related the conditional law of the remaining part after an initial segment to the law in the remaining domain. This property is motivated by interfaces in statistical mechanics models, as the reader will easily understand by considering for example the Ising model on a hexagonal lattice, and an interface which is a boundary between plus and minus spin clusters. In statistical mechanics, this property does not even need the model to be at a critical point. It is remarkable that when we combine the domain Markov property with the conformal invariance anticipated to emerge at the critical point, we obtain a simple classification of the possible random curves. This observation made in [Sch00] is known as the *Schramm's principle* and will be discussed next.

The Schramm's principle (chordal case)

For the techniques of Loewner chains to be applicable, we still have to impose the following regularity assumption on the random curves

- *Loewner regularity*: We assume that the curve $\gamma : [0,T] \to \overline{\Lambda}$ starts from a, i.e. $\gamma(0) = a$, and that the tip of any initial segment $\gamma|_{[0,s]}$ is in the component containing b of the complement $\Lambda \setminus \gamma[0,s]$ of the initial segment, and that the local growth condition is satisfied.

Recall that by conformal invariance, $f_* \mathsf{P}_{(\Lambda;a,b)} = \mathsf{P}_{(f(\Lambda);f(a),f(b))}$, it is enough to describe the random curve in one reference domain, and to push-forward the definition to other domains by conformal maps. For the chordal case it is convenient to choose the reference domain $(\mathbb{H}; 0, \infty)$, and use the chordal Loewner chain in the half-plane to describe the curve.

Assume that our collection of probability measures $(\mathsf{P}_{(\Lambda;a,b)})$ satisfies *conformal invariance*, *domain Markov property* and *Loewner regularity*. Then let γ be a random curve in the half-plane with law $\mathsf{P}_{(\mathbb{H};0,\infty)}$.

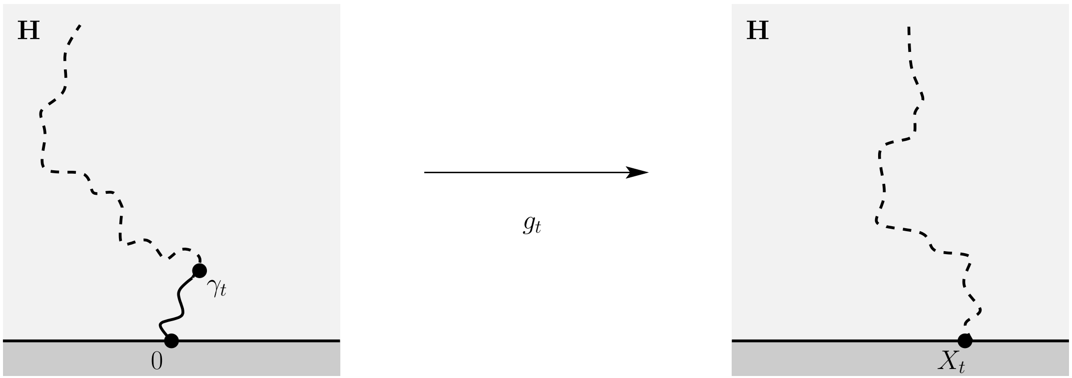


Figure 2.2: The curve is encoded by the Loewner chain $(g_t)_{t\geq 0}$, where g_t is a conformal map from the unbounded component of the complement of $\gamma[0,t]$ to the half-plane.

As in Section 1.1.2, parametrize γ by the half-plane capacity of the initial segments and encode the growth of these initial segments in the Loewner chain. More precisely, let H_t be the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ — the Loewer chain (g_t) then consists of the hydrodynamically normalized conformal maps

$$g_t : H_t \to \mathbb{H}, \qquad g_t(z) \;=\; z + 2t\,z^{-1} + O(z^{-2}).$$

By Loewner regularity, the conformal maps g_t satisfy the Loewner equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2}{g_t(z) - X_t}$$

for some continuous driving function $t \mapsto X_t \in \mathbb{R}$. The random curve γ is encoded by the driving function $t \mapsto X_t$, and we henceforth use the term *driving process* to emphasize the randomness of (X_t).

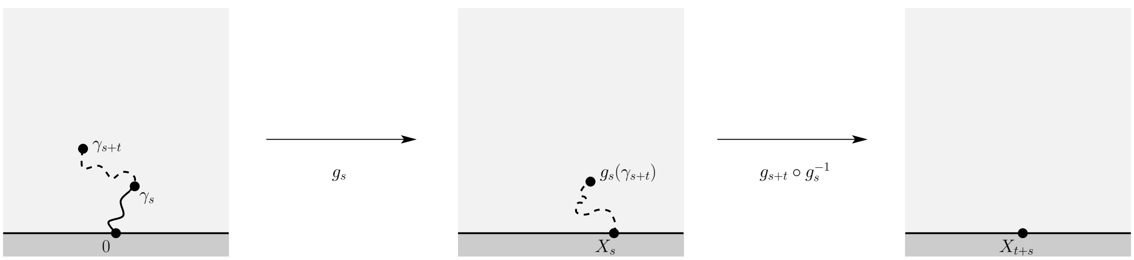


Figure 2.3: To construct the Loewner chain $(\tilde{g}_t)_{t\geq 0}$ for the curve defined by $\tilde{\gamma}_t = g_s(\gamma_{s+t}) - X_s$, we compose the maps of the original Loewner chain.

Consider an initial segment $\gamma|_{[0,s]}$ of the random curve. The Loewner chain to describe the initial segment corresponds to the restriction of the driving process $(X_t)_{t\in[0,s]}$. By the domain

Markov property, the conditional law of the remaining part $\gamma|_{[s,T]}$ given the initial segment is $\mathsf{P}_{(H_s;\gamma(s),\infty)}$. Now note that the map

$$z \;\mapsto\; g_s(z) - X_s$$

is conformal from H_s to $\mathbb{H}$, and such that $\gamma(s) \mapsto 0$ (definition of driving function) and $\infty \mapsto \infty$ (hydrodynamical normalization). Therefore, conditionally on the initial segment, conformal invariance states that the law of the image of $\gamma|_{[s,T]}$ by the map $g_s - X_s$ has the law $\mathsf{P}_{(\mathbb{H};0,\infty)}$. The image curve $\tilde{\gamma}$ is defined by

$$\tilde{\gamma}(t) \;=\; g_s(\gamma(s+t)) - X_s, \qquad t \geq 0.$$

Let $(\tilde{g}_t)$ denote the collection of hydrodynamically normalized conformal maps $\tilde{g}_t : \tilde{H}_t \to \mathbb{H}$, where $\tilde{H}_t$ is the unbounded component of $\mathbb{H} \setminus \tilde{\gamma}[0,t]$. In fact,

$$z \;\mapsto\; g_{s+t}\Big(g_s^{-1}(z+X_s)\Big) - X_s$$

is a conformal map $\tilde{H}_t \to \mathbb{H}$, and it is a matter of simple calculation to verify the normalization

$$\begin{aligned} g_{s+t}\Big(\underbrace{g_s^{-1}(z+X_s)}_{\approx z+X_s-\frac{2s}{z+X_s}+\cdots}\Big) - X_s &= \Big(z+X_s-\frac{2s}{z+X_s}+\cdots\Big) + \frac{2(s+t)}{z+X_s-\frac{2s}{z+X_s}+\cdots} + \cdots - X_s \\ &= \Big(z+X_s-\frac{2s}{z}\Big) + \frac{2(s+t)}{z} - X_s + O(z^{-2}) \\ &= z + \frac{2t}{z} + O(z^{-2}). \end{aligned}$$

We see that not only is the above map hydrodynamically normalized, but also the curve $\tilde{\gamma}$ is still parametrized by capacity. In conclusion the Loewner chain for $\tilde{\gamma}$ is given by

$$\tilde{g}_t(z) \;=\; g_{s+t}\Big(g_s^{-1}(z+X_s)\Big) - X_s.$$

This Loewner chain must satisfy a Loewner's equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}_t(z) \;=\; \frac{2}{\tilde{g}_t(z)-\tilde{X}_t},$$

and indeed from the expression above we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}_t(z) \;=\frac{\mathrm{d}}{\mathrm{d}t}\Big(g_{s+t}\Big(g_s^{-1}(z+X_s)\Big)-X_s\Big) \;=\; \frac{2}{g_{s+t}\Big(g_s^{-1}(z+X_s)\Big)-X_{s+t}} \;=\; \frac{2}{\tilde{g}_t(z)+X_s-X_{s+t}}.$$

We get that the driving process of $(\tilde{g}_t)$ is given by the increment of the driving process of (g_t)

$$\tilde{X}_t \;=\; X_{s+t} - X_s.$$

Moreover, since $\tilde{\gamma}$ has the same law $\mathsf{P}_{(\mathbb{H};0,\infty)}$ as γ, the driving process $(\tilde{X}_t)$ must have the same law as (X_t). Also recall that the considerations so far were done conditionally on the initial segment $\gamma|_{[0,s]}$ or equivalently conditionally on its driving function $(X_t)_{t\in[0,s]}$, so by we see that $(\tilde{X}_t)$ is independent of $(X_t)_{t\in[0,s]}$. The continuous process (X_t) therefore has independent and identically distributed increments, so its law is necessarily that of a multiple of Brownian motion plus linear drift

$$\Big(X_t\Big)_{t\geq 0} \overset{\text{in law}}{\equiv} \Big(\sqrt{\kappa}\,B_t + \alpha\,t\Big)_{t\geq 0}, \qquad \kappa \geq 0,\ \alpha \in \mathbb{R}.$$

However, we now check that only $\alpha = 0$ is consistent with the requirement that $\mathsf{P}_{(\mathbb{H};0,\infty)}$ is invariant under the one parameter family of conformal self maps of $(\mathbb{H};0,\infty)$. These self maps are the scalings of the half-plane, $z \mapsto \lambda z$ for $\lambda > 0$. To map complements of initial segments $\lambda\gamma[0,t]$ of the scaled curve $\lambda\gamma$ hydrodynamically to the half-plane, one uses the map

$$z \;\mapsto\; \lambda\; g_t(z/\lambda).$$

The Laurent expansion

$$\lambda\, g_t(z/\lambda) \;=\; \lambda \left(z/\lambda + \frac{2t}{z/\lambda} + \cdots \right) \;=\; z + \frac{2\lambda^2 t}{z} + \cdots$$

reveals that the correct capacity parametrization of $\lambda\gamma$ is

$$\gamma^{(\lambda)}(t) \;=\; \lambda\, \gamma(\lambda^{-2} t).$$

The capacity parametrized Loewner chain for $\lambda\gamma$ is $(g_t^{(\lambda)})$ with

$$g_t^{(\lambda)}(z) \;=\; \lambda\, g_{\lambda^{-2}t}(z/\lambda).$$

The Loewner equation satisfied by this Loewner chain is obtained by calculating the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^{(\lambda)}(z) \;=\lambda\,\lambda^{-2}\, \frac{2}{g_{\lambda^{-2}t}(z/\lambda) - X_{\lambda^{-2}t}} \;=\; \frac{2}{\lambda\, g_{\lambda^{-2}t}(z/\lambda) - \lambda\, X_{\lambda^{-2}t}} \;=\; \frac{2}{g_t^{(\lambda)}(z) - \lambda\, X_{\lambda^{-2}t}},$$

from which we see that the driving process $(X_t^{(\lambda)})$ of $(g_t^{(\lambda)})$ is

$$X_t^{(\lambda)} \;=\; \lambda\, X_{\lambda^{-2}t}.$$

If $X_t \equiv \sqrt{\kappa}\, B_t + \alpha t$, then

$$X_t^{(\lambda)} \;\equiv\; \lambda\Big(\sqrt{\kappa}\, B_{\lambda^{-2}t} + \alpha\lambda^{-2} t \Big) \;\equiv\; \lambda\, \sqrt{\kappa}\, \sqrt{\lambda^{-2}}\, B_t + \lambda\alpha\lambda^{-2} t \;\equiv\; \sqrt{\kappa}\, B_t + \alpha\lambda^{-1} t.$$

The scaled curve $\lambda\gamma$ would have a different law if $\alpha \neq 0$, so the conformal invariance under self maps of $(\mathbb{H}; 0, \infty)$ requires $\alpha = 0$ and finally

$$X_t \;\equiv\; \sqrt{\kappa}\, B_t.$$

We have obtained a strong classification: if a random conformally invariant chordal curve satisfies domain Markov property (and Loewner regularity), then the curve is the push forward by conformal maps from the half-plane $(\mathbb{H}; 0, \infty)$ of a curve whose Loewner driving process is a multiple of Brownian motion. The requirements we imposed motivated by interfaces in critical models of statistical mechanics characterized the law of a curve up to one parameter κ.

2.1.1 The chordal SLE$_\kappa$

The conclusion obtained by Schramm's principle is that there can be no other conformally invariant chordal random curves with domain Markov property except the ones whose half-plane Loewner chain has driving process $(\sqrt{\kappa}\, B_t)_{t \geq 0}$ for some $\kappa \geq 0$. We thus call the Loewner chain determined by

$$g_0(z) \;= z, \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2}{g_t(z) - X_t}, \qquad\qquad X_t \;=\; \sqrt{\kappa}\, B_t$$

the *chordal Schramm-Loewner evolution with parameter κ in* $(\mathbb{H}; 0, \infty)$, or briefly *chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$.

A priori the chordal SLE$_\kappa$ is a collection of conformal maps $(g_t)_{t \geq 0}$, where $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ is a hydrodynamically normalized map from complements of random hulls $K_t \subset \overline{\mathbb{H}}$, and the hulls are growing: $K_t \subset K_s$ for $t < s$. It is however natural to ask whether there is a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that $K_t = \gamma[0, t]$, or if at least K_t is generated by a curve in the sense that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. The following result answers the latter question in the affirmative.

Theorem 2 (Rohde and Schramm, [RS05]) *For the chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$*, the limits*

$$\gamma(t) \;=\; \lim_{\varepsilon \searrow 0} g_t^{-1}(X_t + \mathrm{i}\varepsilon)$$

exist and depend continuously on $t \geq 0$. We call the curve γ the chordal SLE$_\kappa$ trace in $(\mathbb{H}; 0, \infty)$*. The hulls (K_t) are generated by the trace.*

The proof is somewhat lengthy although not particularly difficult, and in fact the case $\kappa = 8$ needs to be considered separately — it was completed in [LSW04]. The interested reader will find a careful proof in the generic case $\kappa \neq 8$ in [Law10].

Admitting the above result on the chordal SLE trace, we from here on view the chordal SLE_κ as a random curve rather than a Loewner chain. This is certainly closer to the original motivation, and it is worth emphasizing that the curve is the fundamental object, whereas the Loewner chain is merely an artefact resulting from our description of the curve. Figures 2.4 — 2.9 portray simulated chordal SLE_κ traces for a few different values of κ.

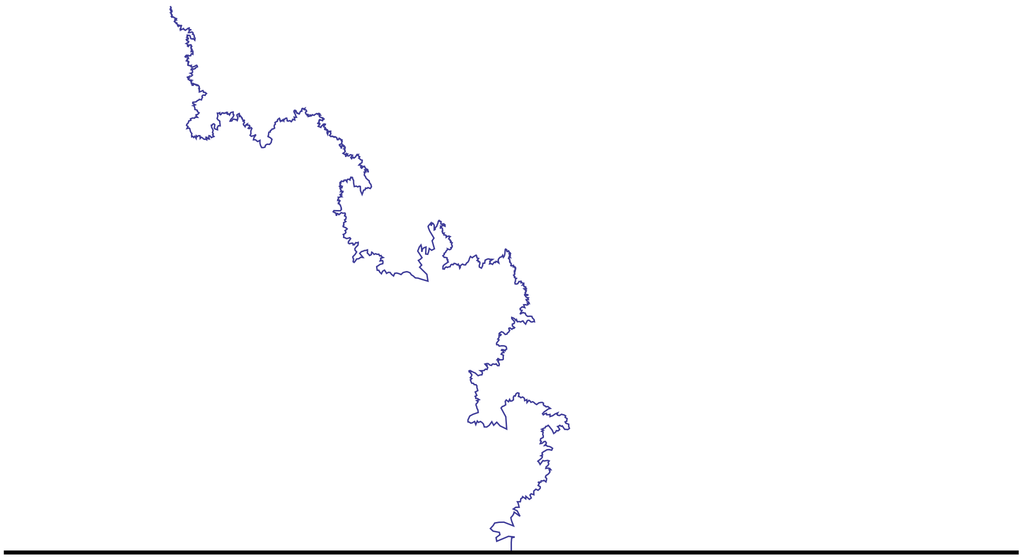


Figure 2.4: Initial segment of a chordal SLE_2 trace in $(\mathbb{H}; 0, \infty)$.

We immediately remark the invariance under conformal self-maps of a domain $(\Lambda; a, b)$. In the following form it directly follows from the scaling calculation we did in the course of establishing the Schramm's principle in the chordal case.

Proposition 1 *The law of chordal SLE_κ in $(\mathbb{H}; 0, \infty)$ is invariant under the scalings $z \mapsto \lambda z$, $\lambda > 0$.*

It is also natural to ask for other properties of SLEs. The following result on the qualitative properties divides the parameter regions of κ to three phases.

Theorem 3 (Rohde and Schramm, [RS05]) *The (trace of the) chordal SLE_κ in $(\mathbb{H}; 0, \infty)$ is transient,*

$$\lim_{t \nearrow \infty} \gamma(t) = \infty,$$

and it has the following properties according to the parameter $\kappa \geq 0$

$0 \leq \kappa \leq 4$: *The trace $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a simple curve, and $\gamma(t) \in \mathbb{H}$ for all $t > 0$.*

$4 < \kappa < 8$: *For any $z \in \mathbb{H}$ almost surely there exists a $t > 0$ such that $z \in K_t$ but $z \notin \gamma[0, t]$, i.e. the trace surrounds (or "swallows") the point z without passing through it. Also $\gamma[0, \infty) \cap \mathbb{R}$ is unbounded.*

$\kappa \geq 8$: *The trace is a space filling curve, $\gamma[0, \infty) = \overline{\mathbb{H}}$, i.e. the trace visits every point of the domain.*

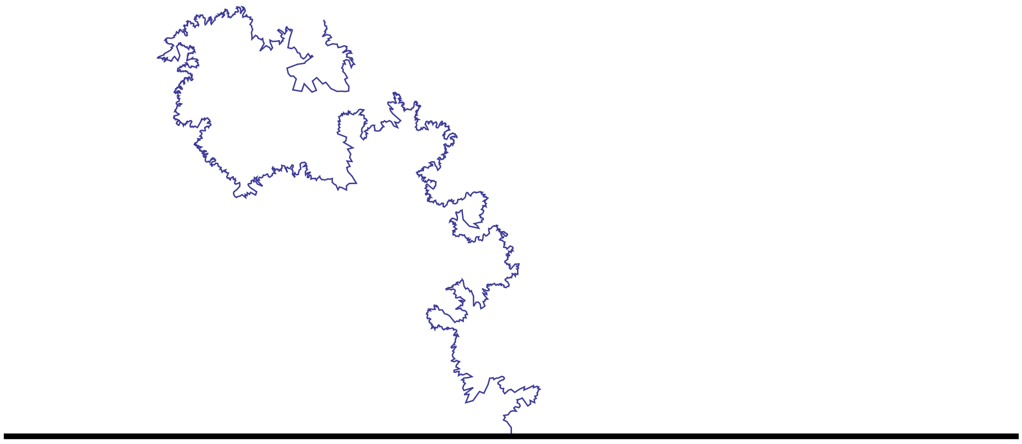


Figure 2.5: Initial segment of a chordal $\mathrm{SLE}_{8/3}$ trace in $(\mathbb{H}; 0, \infty)$.

We will prove some of the statements in the next chapter, the others could be proven by quite similar techniques.

The simulated pictures give some hints about the three phases, although due to the necessary discretization of the curves for the simulation, the pictures have no genuinely different phases. It is also somewhat challenging to reduce numerical errors in simulating SLEs, so from the pictures it might not be clear that phase transitions occur at the precise values of the parameter κ.

One of the most notable quantitative properties of chordal SLE_κ is the fractal dimension. Looking at the simulated pictures, one may already guess that the fractal dimension of the curve increases with the parameter κ.

Theorem 4 (Beffara, [Bef08]) *For $0 \leq \kappa \leq 8$, the Hausdorff dimension of γ, the trace of the chordal SLE_κ, is $1 + \frac{\kappa}{8}$. For $\kappa > 4$, the Hausdorff dimension of ∂K_t, the boundary of the SLE hull, is $1 + \frac{2}{\kappa}$.*

It is not very difficult to obtain the sharp upper bound for the Hausdorff dimension, and in the next chapter we present an argument which leads to the correct value of the dimension. A reasonably accessible and careful proof of the entire result can be found in [Law10].

2.1.2 Other SLEs

It turns out that Schramm's principle works rather well in a few of the simplest situtations besides just the chordal case, notably the following:

- random curves in simply connected domains with three marked boundary points
- random curves in simply connected domains with a marked boundary point and a marked interior point

In each of the two cases, Riemann mapping theorem guarantees that any two domains are conformally equivalent, and in these cases there are no conformal self maps of a domain — all three degrees of freedom are needed to fix the marked points. We may thus expect that the conformal invariance requirement is somewhat less restrictive, and indeed we will find that the classification leaves room for an additional parameter.

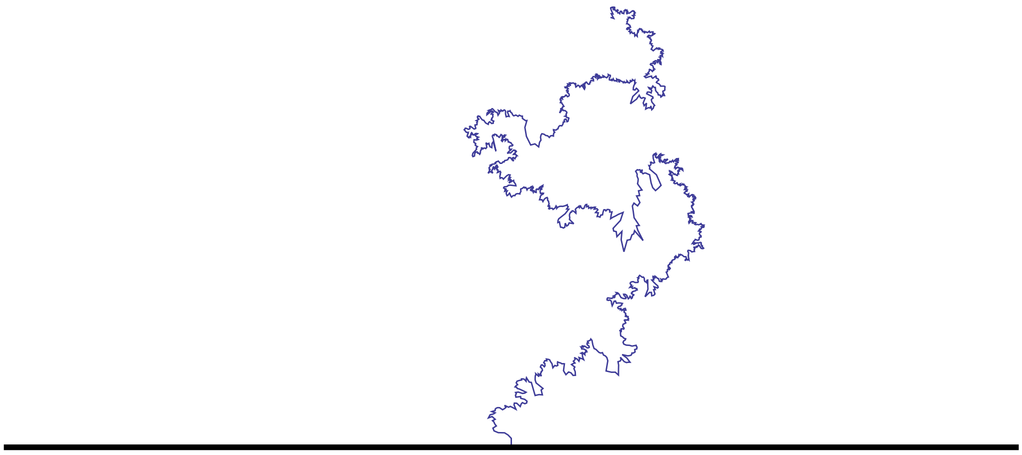


Figure 2.6: Initial segment of a chordal SLE_3 trace in $(\mathbb{H}; 0, \infty)$.

Exercise 4 *Find a Schramm's principle for Loewner regular conformally invariant curves with domain Markov property in the following situation: the domain $\Lambda \subsetneq \mathbb{C}$ is simply connected, the curve start from boundary point $a \in \partial\Lambda$, and the law $\mathsf{P}_{(\Lambda;a,b,c)}$ depends also on two other marked boundary points $b, c \in \partial\Lambda$.* Hint: *It is convenient to work in $(\mathbb{S}; 0, +\infty, -\infty)$, and use a Loewner chain corresponding to Loewner vector fields* $\coth(\frac{z-x}{2})\,\partial_z$.

One could also consider other configurations, such as four or more marked boundary points in simply connected domains, or more marked interior points, or multiply connected domains. In each of these cases, however, there are conformal moduli, i.e. two generic such domains are no longer conformally equivalent. The requirement of conformal invariance then has weaker consequences, and an attempt of classification as above becomes less satisfactory — in applications to statistical mechanics models one needs more input from the model itself for identifying the appropriate random curves.

Figure 2.7: Initial segment of a chordal SLE_4 trace in $(\mathbb{H}; 0, \infty)$.

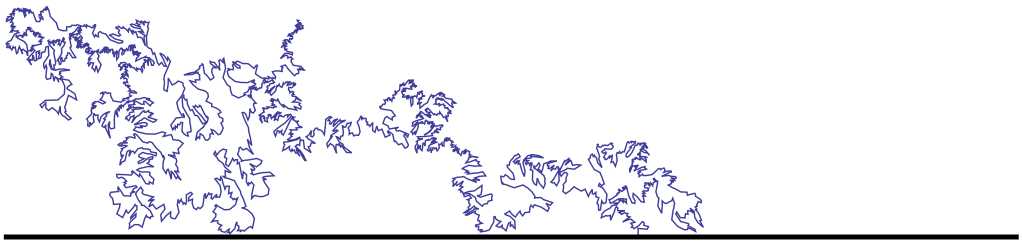
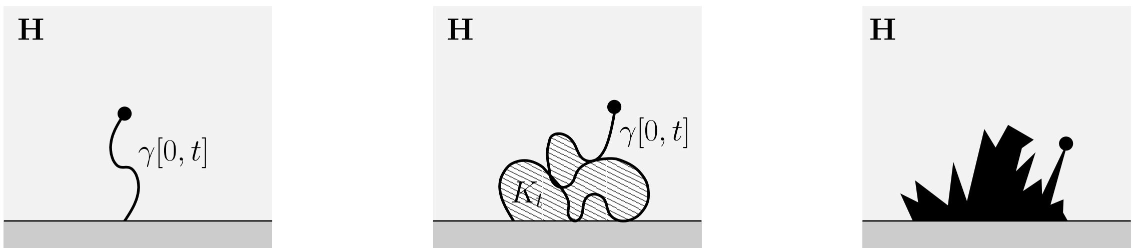


Figure 2.8: Initial segment of a chordal SLE_6 trace in $(\mathbb{H}; 0, \infty)$.

Figure 2.9: Initial segment of a chordal SLE_8 trace in $(\mathbb{H}; 0, \infty)$.



$0 \leq \kappa \leq 4$ $\quad$ $4 < \kappa < 8$ $\quad$ $8 \leq \kappa$

Figure 2.10: According to the value of the parameter κ, SLE is in one of the three qualitatively different phases: the trace γ is either a simple curve, a self-touching curve, or a space filling curve.

Chapter 3

Calculations with Schramm-Loewner evolutions

In this minicourse we present two key techniques for calculating things with SLEs:

- Coordinate changes of SLEs
- Matringales from domain Markov property.

We give a few examples of each of the two techniques, chosen so that the calculations remain simple enough but at the same time illustrate and emphasize some important properties of SLE curves.

It is very common that the two techniques are combined together to derive an interesting property of SLEs, like in the case of the restriction property of chordal $\mathrm{SLE}_{8/3}$ which can be found in almost any introduction to SLEs. Often the calculations also allow for natural interpretations using Girsanov's theorem — a change of drift of the driving process resulting either from coordinate changes or conditioning on an event can be seen as a weighting of the SLE probability measure by a martingale. We will only comment on these interpretations briefly.

3.1 Coordinate changes of SLEs

In this section we consider descriptions of the same random curve by different Loewner chains. We emphasize that the random curve is the fundamental object and its parametrization and Loewner chain description are somewhat arbitrary choices, although certain choices are without a doubt more convenient than others.

The article [?] does several coordinate changes systematically. The same idea and almost identical calculations are fundamental for many different SLE problems, so variations on this theme have appeared in the literature ever since SLEs were introduced.

The chordal SLE in half-plane with another endpoint

We have defined the chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ as the curve which generates the Loewner chain with driving process $(\sqrt{\kappa}\, B_t)_{t\geq 0}$. In any other domain $(\Lambda; a, b)$, the chordal SLE_κ is the image of this curve by a conformal map from $(\mathbb{H}; 0, \infty)$ to $(\Lambda; a, b)$.

Let us consider the case where the domain is still the half-plane $\mathbb{H}$, the starting point still the origin, but the end point is some point $b \in \mathbb{R} \setminus \{0\}$ at finite distance. The chordal SLE_κ in $(\mathbb{H}; 0, b)$ is clearly a Loewner regular curve (up to the first time it disconnects b from infinity), so we can give a description of it by a chordal Loewner chain.

So, let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be the chordal SLE_κ trace in $(\mathbb{H}; 0, \infty)$ parametrized by capacity as above, and let $\hat{\gamma}(t) = \mu(\gamma(t))$, where μ is a conformal map from $(\mathbb{H}; 0, \infty)$ to $(\mathbb{H}; 0, b)$. Such conformal maps are Möbius transformations, and there is a one parameter family of them:

$$\mu(z) = \frac{b\, z}{z - s}$$

where the parameter $s \in \mathbb{R} \setminus \{0\}$ is the point whose image under μ is infinity. We will use a Loewner chain that fixes infinity to describe the image curve $\hat{\gamma} = \mu \circ \gamma$, so in a sense we are observing the original SLE curve in $(\mathbb{H}; 0, \infty)$ from the point s.

Let $(g_t)_{t \geq 0}$ be the Loewner chain defining the chordal SLE$_\kappa$ in $(\mathbb{H}; 0, \infty)$,

$$g_0(z) \;=\; z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2}{g_t(z) - X_t}, \qquad X_t \;=\; \sqrt{\kappa}\, B_t.$$

We want to find the Loewner chain that describes $\hat{\gamma} = \mu \circ \gamma$, the chordal SLE$_\kappa$ in $(\mathbb{H}; 0, b)$. To this end, let $\hat{g}_t$ be the hydrodynamically normalized conformal map from the unbounded component $\hat{H}_t$ of $\mathbb{H} \setminus \hat{\gamma}[0,t]$ to $\mathbb{H}$. Note that $\hat{\gamma}$ is not yet parametrized by capacity, but it is not difficult to see that it would only take a differentiable change of parametrization to achieve that. Let us denote by s_t the half-plane capacity of the hull generated by $\hat{\gamma}[0,t]$, so that $\hat{g}_t(z) = z + 2 s_t\, z^{-1} + O(z^{-2})$. Then by Loewner regularity of the image curve $\hat{\gamma}$, until the time that $\hat{\gamma}$ disconnects b from ∞ we have

$$\hat{g}_0(z) \;=\; z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \hat{g}_t(z) \;=\; \frac{2\, \dot{s}_t}{\hat{g}_t(z) - \xi_t}$$

for some driving process $(\xi_t)_{t \geq 0}$, and with $\dot{s}_t = \frac{\mathrm{d}}{\mathrm{d}t} s_t$ the speed of capacity growth of $\hat{\gamma}$.

We already have at our disposal the conformal map $g_t \circ \mu^{-1} : \hat{H}_t \to \mathbb{H}$. The hydrodynamically normalized conformal map $\hat{g}_t : \hat{H}_t \to \mathbb{H}$ is obtained by post-composing with an appropriate self map μ_t of the half-plane,

$$\hat{g}_t \;=\; \mu_t \circ g_t \circ \mu^{-1}.$$

One could give an explicit expression for the time dependent Möbius transformation μ_t, but it turns out to be not necessary. We note that the driving process (ξ_t) is the image under the Loewner chain $(\hat{g}_t)$ of the tip of $\hat{\gamma}$, or alternatively

$$\xi_t \;=\; \hat{g}_t\big(\hat{\gamma}(t)\big) \;=\; \big(\mu_t \circ g_t \circ \mu^{-1}\big)\big(\mu(\gamma(t))\big) \;=\; \mu_t(X_t).$$

Also since

$$\mu_t \;=\; \hat{g}_t \circ \mu \circ g_t^{-1},$$

we can calculate the time derivative of $\mu_t(z)$. We just recall the Loewner equation for $\hat{g}_t$, and observe that the time derivative of g_t^{-1} is easily read from

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}(z) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(g_t(g_t^{-1}(z))\Big) \;=\; \frac{2}{g_t(g_t^{-1}(z)) - X_t} + g_t'(g_t^{-1}(z)) \Big(\frac{\mathrm{d}}{\mathrm{d}t} g_t^{-1}(z)\Big),$$

with the result

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^{-1}(z) \;=\; \frac{-2\,(g_t^{-1})'(z)}{z - X_t}.$$

Now we calculate the time derivative of μ_t as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu_t(z) \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\hat{g}_t(\mu(g_t^{-1}(z)))\Big) \\ &=\; \frac{2\,\dot{s}_t}{\hat{g}_t(\mu(g_t^{-1}(z))) - \xi_t} + (\hat{g}_t \circ \mu)'(g_t^{-1}(z))\; \frac{-2\,(g_t^{-1})'(z)}{z - X_t} \\ &=\; \frac{2\,\dot{s}_t}{\mu_t(z) - \xi_t} - \frac{2\,\mu_t'(z)}{z - X_t}. \end{aligned}$$

The Möbius transformation $\mu_t : \mathbb{H} \to \mathbb{H}$, and its time derivative as well, is regular at the point X_t on the boundary (the only pole of μ_t is at the point $g_t(s)$, so that $\hat{g}_t = \mu_t \circ g_t \circ \mu^{-1}$ fixes infinity). Therefore, the poles at $z \to X_t$ of the two terms in $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(z)$ must cancel. We do a Laurent expansion for the first term, keeping in mind that $\mu_t(X_t) = \xi_t$,

$$\begin{aligned} \frac{2\,\dot{s}_t}{\mu_t(z) - \xi_t} \;&=\; \frac{2\,\dot{s}_t}{\Big(\xi_t + \mu_t'(X_t)\,(z - X_t) + \frac{1}{2}\mu_t''(X_t)\,(z - X_t)^2 + \cdots\Big) - \xi_t} \\ &=\; \frac{2\,\dot{s}_t}{\mu_t'(X_t)\,(z - X_t)} - \frac{\dot{s}_t\,\mu_t''(X_t)}{\mu_t'(X_t)^2} + O(z - X_t). \end{aligned}$$

The second term is even easier

$$-\frac{2\,\mu_t'(z)}{z-X_t} \;=\; \frac{-2\,\mu_t'(X_t)}{z-X_t} - 2\mu_t''(X_t) + O(z-X_t).$$

For the poles to cancel, we must have

$$\dot{s}_t \;=\; \mu_t'(X_t)^2.$$

This is of course intuitive. On the one hand, $\dot{s}_t$ is the speed of capacity growth of the curve $\hat{\gamma}$ at time t. On the other hand, a small piece of the curve $\hat{\gamma}[t, t+\Delta t]$ becomes, after mapping to the half-plane by $\hat{g}_t$, the image of $g_t(\gamma[t, t+\Delta t])$ under μ_t. But $g_t(\gamma[t, t+\Delta t])$ is a small piece of curve in $\mathbb{H}$ of capacity Δt, and it is located near X_t. So μ_t essentially scales this piece by the factor $\mu_t'(X_t)$ and the image has capacity approximately $\mu_t'(X_t)^2\,\Delta t$, which is the asserted capacity growth $s_{t+\Delta t} - s_t$.

We made the expansions at $z \to X_t$ of the two terms in $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(z)$ up to constant terms, so we immediately read the time derivative of μ_t at the point X_t,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mu_t\Big)(X_t) \;=\; -3\,\mu_t''(X_t).$$

This facilitates the determination of the driving process (ξ_t) of the Loewner chain $(\hat{g}_t)$ since $\xi_t = \mu_t(X_t)$, as we observed earlier. Now, recalling that $\mathrm{d}X_t = \sqrt{\kappa}\,\mathrm{d}B_t$, the Itô derivative of ξ_t is

$$\begin{aligned}\mathrm{d}\xi_t \;&= \mathrm{d}\big(\mu_t(X_t)\big) \;=\; \mu_t'(X_t)\,\sqrt{\kappa}\,\mathrm{d}B_t + \frac{\kappa}{2}\mu_t''(X_t)\,\mathrm{d}t + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mu_t\Big)(X_t)\,\mathrm{d}t \\ &= \mu_t'(X_t)\,\sqrt{\kappa}\,\mathrm{d}B_t + \frac{\kappa-6}{2}\mu_t''(X_t)\,\mathrm{d}t.\end{aligned}$$

We may further remark that any Möbius transformation ν has the property

$$\frac{\nu'(z)^2}{\nu''(z)} \;=\; \frac{1}{2}\,\big(\nu(z) - \nu(\infty)\big).$$

Applied to μ_t at X_t, noting $\mu_t(\infty) = \hat{g}_t(b)$, this gives

$$\frac{\mu_t''(X_t)}{\mu_t'(X_t)^2} \;=\; \frac{2}{\xi_t - \hat{g}_t(b)},$$

which allows us to simplify the Itô derivative of ξ_t to

$$\mathrm{d}\xi_t \;=\; \sqrt{\kappa\,\dot{s}_t}\,\mathrm{d}B_t + \frac{\kappa-6}{\xi_t - \hat{g}_t(b)}\,\dot{s}_t\,\mathrm{d}t.$$

In order to have a standard Loewner chain description of $\hat{\gamma}$, the chordal SLE_κ in $(\mathbb{H}; 0, b)$, we should use $s = s_t$ as the time parameter. Denote by $s \mapsto t_s$ the inverse function of $t \mapsto s_t$. Then the Loewner equation takes the usual form

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{g}_{t_s}(z) \;=\; \frac{2}{\hat{g}_{t_s}(z) - \xi_{t_s}}$$

and the change of time parametrization of the driving process leads to

$$\mathrm{d}\xi_{t_s} \;=\; \sqrt{\kappa}\,\mathrm{d}\hat{B}_s + \frac{\kappa-6}{\xi_t - \hat{g}_t(b)}\,\mathrm{d}s,$$

where $(\hat{B}_s)_{s\geq 0}$ is a standard Brownian motion with respect to the time parameter s. This displays that the change of the chordal SLE endpoint to b excerts a drift on the driving process, whose strength is inversely proportional to the conformal distance of the tip and the endpoint. The sign and strength of the drift depend on κ, and at $\kappa = 6$ the additional drift vanishes.[1]

[1]This particular phenomenon at $\kappa = 6$ gets a natural interpretation from a percolation result of Smirnov. The chordal SLE_6 is the scaling limit of exploration path of critical percolation — and the exploration path doesn't feel where its declared endpoint is.

The Loewner chain $(\hat{g}_{t_s})_{s\geq 0}$ is of the form that is usually taken as definition of the SLE variant $\mathrm{SLE}_\kappa(\rho)$ in the domain $(\mathbb{H}; 0, b, \infty)$, with the particular value $\rho = \kappa - 6$ here. Below we will discuss the Schramm's principle applied to simply connected domains with three marked boundary points, and conclude that the most general (Loewner regular) conformally invariant random curves with domain Markov property are $\mathrm{SLE}_\kappa(\rho)$, for $\kappa \geq 0$ and $\rho \in \mathbb{R}$. In view of this fact, the result of the coordinate change had to be of this form.

The process $\mathrm{SLE}_\kappa(\kappa - 6)$ is also instrumental for the construction of so called conformal loop ensembles via an exploration tree, but for this purpose the process has to be continued in a slightly nontrivial fashion beyond the first time that the curve disconnects the target point b from the observation point ∞. This is beyond the scope of the present minicourse, but the interested reader may consult the article [She09] for further information.

SLEs with three marked boundary points

In an earlier exercise, the following version of Schramm's principle was considered. To each simply connected domain $\Lambda \subsetneq \mathbb{C}$ and three boundary points $a, b, c \in \partial\Lambda$ one associates a probability measure $\mathsf{P}_{(\Lambda;a,b,c)}$ on Loewner regular curves starting from a and ending on the arc $\widehat{bc}$, and such that conformally invariance holds in the sense that $f_* \mathsf{P}_{(\Lambda;a,b,c)} = \mathsf{P}_{(f(\Lambda);f(a),f(b),f(c))}$ for f a conformal map. The classification result is that such curve in $(\mathbb{S}; 0, +\infty, -\infty)$ must be described by a Loewner chain

$$h_0(z) \;=\; z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} h_t(z) \;=\; \coth\Big(\frac{h_t(z) - V_t}{2}\Big), \qquad V_t \;=\; \sqrt{\kappa}\, B_t + \alpha\, t,$$

for some $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. In other domains the curve can be defined by conformal transport, as the image of the curve in $(\mathbb{S}; 0, +\infty, -\infty)$ under a conformal map $f : \mathbb{S} \to \Lambda$ such that $f(0) = a$, $f(+\infty) = a$, $f(-\infty) = c$.

For easier comparison with the chordal SLE, let us take the curve in the upper half-plane so that it starts from the origin and one of the marked points is at infinity. To obtain the curve in $(\mathbb{H}; 0, \infty, c)$, where $c < 0$, we use the conformal map f from $\mathbb{S}$ to $\mathbb{H}$ such that $f(0) = 0$, $f(+\infty) = \infty$ and $f(-\infty) = c$.

A formula for that map is

$$f(z) \;=\; |c|(e^z - 1).$$

Let $\eta : [0, \infty) \to \overline{\mathbb{S}}$ be the curve in $(\mathbb{S}; 0, +\infty, -\infty)$ and consider the image $\gamma(t) = f(\eta(t))$. The component of $\mathbb{S} \setminus \eta[0, t]$ which contains both $\pm\infty$ is denoted by S_t and the Loewner chain $(h_t)_{t\geq 0}$ consists of conformal maps $h_t : S_t \to \mathbb{S}$ normalized so that $h_t(z) - z \to \pm t$ as $z \to \pm\infty$. The curve γ is Loewner regular, too, and $H_t = f(S_t)$ is the component of $\mathbb{H} \setminus \gamma[0, t]$ which contains infinity (and c). Again, the curve γ as we define it is not parametrized by half-plane capacity, but it takes a C^1 reparametrization to achieve this. Denote again by s_t half the half-plane capacity of the hull generated by $\gamma[0, t]$, so that if $g_t : H_t \to \mathbb{H}$ is the hydrodynamical conformal map, then $g_t(z) = z + 2 s_t z^{-1} + O(z^{-2})$. The maps (g_t) satisfy the Loewner flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2\,\dot{s}_t}{g_t(z) - X_t},$$

where $\dot{s}_t = \frac{\mathrm{d}}{\mathrm{d}t} s_t$ is half the speed of capacity growth and X_t is the image of the position of local growth

$$X_t \;=\; g_t(\gamma(t)) \;=\; g_t(f(\eta(t))).$$

We have at our disposal one conformal map from H_t to $\mathbb{H}$, namely the composition $f \circ h_t \circ f^{-1}$, but it is not hydrodynamically normalized. The hydrodynamically normalized maps can be obtained by post-composing with the appropriately chosen conformal self map of $\mathbb{H}$,

$$g_t \;=\; \mu_t \circ f \circ h_t \circ f^{-1},$$

where $\mu_t : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation. In this situation μ_t preserves infinity, so we can write $\mu_t(z) = a_t\, z + b_t$, and it turns out not to be necessary to write down a_t and b_t explicitly (although it is not difficult to do so, and the reader should perhaps nevertheless do that as an exercise). For brevity, let us also denote by

$$\varphi_t \;=\; \mu_t \circ f \;=\; g_t \circ f \circ h_t^{-1}$$

the conformal map $\mathbb{S} \to \mathbb{H}$ which is important for us at time t. This map in particular gives us the driving process of γ,

$$X_t \;=\; \varphi_t(V_t).$$

We will next calculate the time derivative of the map φ_t. For this purpose we need the time derivative of h_t^{-1}, and we leave it as an easy exercise for the reader to check that

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^{-1}(z) \;=\; -(h_t^{-1})'(z)\;\coth\Big(\frac{z-V_t}{2}\Big).$$

Using this and the Loewner flow equation for g_t, we write the time derivative we are interested in as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(g_t(f(h_t^{-1}(z)))\Big) \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}t}g_t\Big)(f(h_t^{-1}(z))) + (g_t \circ f)'(h_t^{-1}(z)))\,\Big(\frac{\mathrm{d}}{\mathrm{d}t}h_t^{-1}\Big)(z) \\
&= \frac{2\,\dot{s}_t}{g_t(f(h_t^{-1}(z))) - X_t} - (g_t \circ f)'(h_t^{-1}(z)))\;(h_t^{-1})'(z)\;\coth\Big(\frac{z-V_t}{2}\Big) \\
&= \frac{2\,\dot{s}_t}{\varphi_t(z) - X_t} - \varphi_t'(z)\;\coth\Big(\frac{z-V_t}{2}\Big).
\end{aligned}$$

Taylor expansion of the two terms at $z = V_t$ like before gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) \;=\; \frac{1}{z-V_t}\Big(\frac{2\,\dot{s}_t}{\varphi_t'(V_t)} - 2\,\varphi_t'(V_t)\Big) - \frac{\dot{s}_t}{\varphi_t'(V_t)^2} - 2\,\varphi_t''(V_t) + O(z-V_t).$$

The maps φ_t as well as their derivatives are regular at the boundary point V_t, so we require the pole to cancel, and obtain the equation

$$\dot{s}_t \;=\; \varphi_t'(V_t)^2,$$

which again is the intuitive property resulting from the change of capacity under the approximate scaling that φ_t does in neighborhoods of V_t. So we have an expression for the (explicit) time derivative of φ_t at the point V_t

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big)(z) \;=\; -3\,\varphi_t''(V_t) + O(z-V_t).$$

We are in a position to compute the increment of $X_t = \varphi_t(V_t)$ by Itô's formula, recalling also $\mathrm{d}V_t = \sqrt{\kappa}\,\mathrm{d}B_t + \alpha\,\mathrm{d}t$. The result is

$$\begin{aligned}
\mathrm{d}X_t \;&=\; \mathrm{d}\Big(\varphi_t(V_t)\Big) \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big)(V_t)\;\mathrm{d}t + (\sqrt{\kappa}\;\mathrm{d}B_t + \alpha\;\mathrm{d}t)\;\varphi_t'(V_t) + \frac{\kappa}{2}\,\mathrm{d}t\;\varphi_t''(V_t) \\
&= \sqrt{\kappa\,\dot{s}_t}\;\mathrm{d}B_t + \Big(\frac{\kappa-6}{2}\,\frac{\varphi_t''(V_t)}{\varphi_t'(V_t)^2} + \alpha\frac{1}{\varphi_t'(V_t)}\Big)\,\dot{s}_t\;\mathrm{d}t.
\end{aligned}$$

We leave it for the reader to verify using the expressions we have found so far, that

$$\frac{\varphi_t''(V_t)}{\varphi_t'(V_t)^2} \;=\; \frac{1}{X_t - g_t(c)} \;=\; \frac{1}{\varphi_t'(V_t)}.$$

Then we may perform a time change to the the half-plane capacity time parameter s, under which we have the ordinary

$$\frac{\mathrm{d}}{\mathrm{d}s}g_{t_s}(z) \;=\; \frac{2}{g_{t_s}(z) - X_{t_s}}.$$

The driving process becomes

$$\mathrm{d}(X_{t_s}) \;=\; \sqrt{\kappa}\, \mathrm{d}\tilde{B}_s + \Big(\frac{\kappa - 6}{2} + \alpha\Big) \frac{1}{X_{t_s} - g_{t_s}(c)}\, \mathrm{d}s.$$

The driving process of the image curve is the one that defines $\mathrm{SLE}_\kappa(\rho)$ in $(\mathbb{H}; 0, \infty, c)$, with $\rho = \frac{\kappa-6}{2} + \alpha$. We have thus shown by a Schramm's principle (the exercise with three marked boundary points) and a coordinate change that the most general Loewner regular conformally invariant random curve which satisfies the domain Markov property and depends on three boundary points of a simply connected domain, is $\mathrm{SLE}_\kappa(\rho)$, for $\kappa \geq 0$ and $\rho \in \mathbb{R}$.

As a notable example, the curve η would have been a chordal SLE_κ in $(\mathbb{S}; 0, +\infty)$ if $\alpha = \frac{6-\kappa}{2}$. Reflecting the sign of the drift α, the chordal SLE_κ in $(\mathbb{S}; 0, -\infty)$ would have $\alpha = \frac{\kappa-6}{2}$, and coming back to the half-plane picture, the chordal SLE_κ in $(\mathbb{H}; 0, c)$ corresponds to $\rho = \kappa - 6$ as we already found before.

3.2 SLE martingales constructed by domain Markov property

One of the most important ways of calculating things with SLEs consists in finding a martingale whose end value is the quantity of interest. The domain Markov property provides a natural way of constructing martingales that compute relevant quantities. We will illustrate this technique in two example cases. The first example is a computation of the probability that the chordal SLE_κ touches a given boundary arc. This case explains first of all why $\kappa = 4$ is the point of phase transition from simple curves to self-touching curves, and it also gives a certain crossing probability that is interesting in the statistical mechanics models. The second example concerns the dimension of the SLE trace. Here we in fact only state a property which gives an upper bound for the Hausdorff dimension, and furthermore we only give a heuristic derivation, which could be made rigorous by slightly altering the definitions and putting in a little bit of extra work. The rigorous derivation can be found in the literature, but the heuristic derivation is shorter and at least as enlightening as the proper one.

Boundary visits of chordal SLE

Let $\gamma^{(\Lambda;a,b)}$ denote the chordal SLE_κ trace in the domain Λ from $a \in \partial\Lambda$ to $b \in \partial\Lambda$. Our goal is to find an expression for

$$\mathsf{P}\Big[\gamma^{(\Lambda;a,b)} \cap A \neq \emptyset\Big],$$

where $A \subset \partial\Lambda \setminus \{a, b\}$ is an arc of the boundary of the domain. By conformal invariance, it is sufficient to find an answer in the reference domain $(\mathbb{H}; 0, \infty)$.

The phase transition at $\kappa = 4$

The first thing we show is that the chordal SLE_κ only touches the boundary when $\kappa > 4$. If the chordal SLE_κ trace γ in $(\mathbb{H}; 0, \infty)$ touches the boundary at a point $x \in (0, \infty)$, then all the points $x' \in (0, x)$ are disconnected from infinity by the curve, and they belong to the SLE hull after the time s such that $\gamma(s) = x$. Furthermore, by the scale invariance stated in Proposition 1, either all $x' > 0$ have a positive probability to become part of the hull, or no $x' > 0$ ever becomes a part of the hull. The question of whether the trace can touch the boundary at any other point but the starting point $a = 0$ and the end point $b = \infty$, is equivalent to whether the boundary points can become a part of the hull.

Recall that the hull K_s is defined as the set of points $z \in \overline{\mathbb{H}}$ such that the Loewner equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_t \;=\; \frac{2}{Z_t - X_t}$$

with initial condition

$$Z_0 \;=\; z \in \overline{\mathbb{H}}$$

has no solution up to time s, i.e. that the denominator $D_t = Z_t - X_t$ becomes zero before the times s (or at least its values accumulate at 0). The denominator is governed by the Itô stochastic differential equation

$$\mathrm{d}D_t \;=\; \frac{2}{D_t}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_t.$$

This is in fact just a time change of the familiar Bessel process: if $t(u) = u/\kappa$, then with respect to the time parameter u we have

$$\mathrm{d}D_{t(u)} \;=\; \frac{2/\kappa}{D_{t(u)}}\,\mathrm{d}u + \mathrm{d}\tilde{B}_u,$$

where $(\tilde{B}_u)_{u\geq 0}$ is a standard Brownian motion: $\tilde{B}_u = -\sqrt{\kappa}\,B_{u/\kappa}$. Recall that the Bessel process (β_t) of dimension d is defined by the Itô stochastic differential equation

$$\mathrm{d}\beta_t \;=\; \frac{(d-1)/2}{\beta_t}\,\mathrm{d}t + \mathrm{d}B_t,$$

and for integer d it corresponds to the absolute value of the d-dimensional Brownian motion. The Bessel process hits the origin in finite time (when started away from the origin) if and only if $d < 2$. Comparing the equations we equate $d = 1 + 4/\kappa$, and correspondingly the denominator process D_t hits zero if and only if $\kappa > 4$.

We have shown that the chordal SLE_κ trace touches the boundary if and only if $\kappa > 4$. The reader may judge to which extent the pictures 2.4 – 2.9 plausibly illustrate this phenomenon. Let us furthermore remark that SLE's touching the boundary is equivalent to self touching of the curve. Indeed, suppose that the chordal SLE_κ trace γ in $(\mathbb{H}; 0, \infty)$ has a double point $\gamma(t_1) = \gamma(t_2)$ for $0 \leq t_1 < t_2$. Pick $s \in (t_1, t_2)$, and consider conditioning on $\gamma[0, s]$. By the domain Markov property, the conditional law of $\gamma[s, \infty)$ is the law of chordal SLE_κ in $(H_t; \gamma(s); \infty)$. But if $\gamma(t_2) = \gamma(t_1)$, then the point $\gamma(t_2)$ on the curve $\gamma[s, \infty)$ is not in the interior of the domain H_t, which means that the chordal SLE_κ touches the boundary of its domain (by conformal invariance it doesn't matter in which domain this happens). Thus, admitting the existence of the chordal SLE trace, we have shown the first phase transition stated in Theorem **??**: for $\kappa \leq 4$ the trace is a simple curve which doesn't touch the boundary of the domain, and for $\kappa > 4$ the trace has double points and touches the boundary.

The martingale and the probability to touch a boundary interval

Let us then compute the probability for the chordal SLE_κ, $\kappa > 4$, to touch a boundary arc. The equivalent question in the half-plane is to compute

$$P(z^-, z^+) \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap [z^-, z^+] \neq \emptyset\Big],$$

where $0 < z^- < z^+$, say (intervals on the negative real axis are handled similarly). The technique to do so relies on finding a martingale whose end value is the indicator of the event that the boundary interval is touched.

The conditional expected values of any random variable, conditioned on the initial segments $\gamma^{(\mathbb{H};0,\infty)}[0, t]$, $t \geq 0$, constitute a martingale by construction. In particular, the conditional probabilities

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap [z^-, z^+] \neq \emptyset \;\Big|\; \gamma^{(\mathbb{H};0,\infty)}[0, t]\Big]$$

form a martingale (M_t).

By the domain Markov property, given the initial segment $\gamma^{(\mathbb{H};0,\infty)}[0, t]$, the remaining part $\gamma^{(\mathbb{H};0,\infty)}[t, \infty)$ of the curve has the law of the chordal SLE_κ trace in $(H_t; \gamma(t), \infty)$, so we can write

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(H_t;\gamma(t),\infty)} \cap [z^-, z^+] \neq \emptyset\Big].$$

Furthermore, we may use the conformal map $z \mapsto g_t(z) - X_t$ from $(H_t; \gamma(t), \infty)$ back to the reference domain $(\mathbb{H}; 0, \infty)$, and by conformal invariance of SLE the image curve $g_t \circ \gamma|_{[t,\infty)} - X_t$ has the law of a chordal SLE_κ in $(\mathbb{H}; 0, \infty)$. The remaining part $\gamma[t, \infty)$ touches the interval $[x_1, x_2]$ if and only if the image curve touches the interval $[g_t(z^-) - X_t, g_t(z^+) - X_t]$, and thus the martingale reads simply

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \,\cap\, [g_t(z^-) - X_t, g_t(z^+) - X_t] \;\neq\; \emptyset\Big] \;=\; P(g_t(z^-) - X_t, g_t(z^+) - X_t).$$

Thus, the domain Markov property and conformal invariance allowed us to express the martingale as a function of two stochastic processes, essentially the Loewner flows of the points z^- and z^+ (translated by an amount determined by the driving process X_t). In fact Proposition 1, the scale invariance of the chordal SLE_κ, allows us to simplify further, since the probability of touching an interval $[z^-, z^+]$ can only depend on the ratio z^-/z^+

$$P(z^-, z^+) \;=\; p\Big(\frac{z^-}{z^+}\Big).$$

For the moment, let us suppose that the function $p : (0,1) \to [0,1]$ in the expression

$$M_t \;=\; p\Big(\frac{g_t(z^-) - X_t}{g_t(z^+) - X_t}\Big)$$

is nice enough, say twice continuously differentiable, so that we can apply Itô's formula. These types of assumptions become justified in the end of the calculation in an almost automatical manner. Recall that the numerator and denominator in the ratio individually follow time changed Bessel processes, both driven by the same Brownian motion

$$\mathrm{d}Z_t^\pm \;=\; \frac{2}{Z_t^\pm} - \sqrt{\kappa}\,\mathrm{d}B_t.$$

Computing the Itô derivative of $M_t = p(Z_t^-/Z_t^+)$ is routine, and the result is

$$\mathrm{d}M_t \;=\mathrm{d}t\,\Big(\frac{Z_t^- - Z_t^+}{2\,(Z_t^+)^2\,Z_t^-}\,\Big((-4 + (2\kappa - 4)r_t)\,p'(r_t) - \kappa\,r_t\,(1 - r_t)\,p''(r_t)\Big)\Big) + \mathrm{d}B_t\Big(\cdots\Big),$$

where we have denoted the ratio by $r_t = Z_t^-/Z_t^+$. We only care about the $\mathrm{d}t$ term, because if (M_t) is to be a martingale, this term has to vanish. Now requiring the $\mathrm{d}t$ term to vanish for generic values of the ratio r_t amounts to the differential equation

$$(-4 + r(2\kappa - 4))\,p'(r) - \kappa(1 - r)r\,p''(r) \;=\; 0.$$

Integrate to get

$$p'(r) \;=\; \text{const.} \,\times\, r^{-\frac{4}{\kappa}}\,(1 - r)^{2\frac{4-\kappa}{\kappa}},$$

and thus

$$p(r) \;=\; c_1 + c_2 \int_r^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u.$$

The differential equation has a two dimensional solution space, but of course there are boundary conditions that the correct solution must satisfy. When the ratio $r = z^-/z^+$ tends to zero, scale invariance tells us that the probability that the SLE curve touches the interval must become one if the curve is ever to touch the boundary. Similarly, when r tends to one, the interval shrinks to a point and the probability to touch the interval should tend to zero (unless the curve visits each point, which is indeed what would happen for $\kappa \geq 8$). The constants are thus determined

$$c_1 \;=\; 0, \qquad c_2 \;=\; \frac{1}{\int_0^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u},$$

although the latter only makes sense if the integral is convergent, which indeed requires $\kappa > 4$ (for convergence at $u = 0$) and $\kappa < 8$ (for convergence at $u = 1$), and we have found

$$p(r) \;=\; \frac{4\,\sqrt{\pi}}{2^{8/\kappa}\,\Gamma(\frac{8-\kappa}{2\kappa})\,\Gamma(\frac{\kappa-4}{\kappa})} \int_r^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u.$$

The standard way to finish the argument, and in particular to get rid of the so far unjustified assumption that p is twice continuously differentiable, is to reverse the logic of the above reasoning. We define p by the formula we just found, and then notice that the process

$$t \mapsto p\Big(\frac{Z_t^-}{Z_t^+}\Big)$$

is a local martingale (by Itô's formula) — not surprisingly as this is what we believe M_t is! The process is furthermore bounded (the values of p are between zero and one), so it is a uniformly integrable martingale up to any stopping time until which both $Z_t^\pm$ are well defined. Take τ_{z^-} to be the stopping time at which z^- becomes a part of the hull — this is also the first time at which the SLE trace γ touches the interval $[z^-, \infty)$. There are two possibilities regarding z^+. If the point $\gamma(\tau_{z^-})$ is on the interval $[z^-, z^+]$, then the point z^+ does not become a part of the hull yet, and the ratio Z_t^-/Z_t^+ tends to zero (the denominator remains non-zero, while the numerator tends to zero). In this case the value of our uniformly integrable martingale tends to $p(0) = 1$. If, however, the point $\gamma(\tau_{z^-})$ is on the interval (z^+, ∞), then easy estimates of harmonic measure show that the ratio Z_t^-/Z_t^+ tends to one. In this case the value of our uniformly integrable martingale tends to $p(1) = 0$. We conclude that the endvalue of the uniformly integrable matringale is the indicator of the event we are interested in

$$\lim_{t \nearrow \tau_{z^-}} p\Big(\frac{Z_t^-}{Z_t^+}\Big) = \begin{cases} 1 & \text{if } \gamma \text{ touches } [z^-, z^+] \\ 0 & \text{if } \gamma \text{ doesn't touch } [z^-, z^+] \end{cases},$$

so with the optional stopping theorem we have derived the probability we were interested in

$$\mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \text{ touches } [z^-, z^+]\Big] = \mathsf{E}\Big[p\Big(\frac{Z_{\tau_{z^-}}^-}{Z_{\tau_{z^+}}^+}\Big)\Big] = \mathsf{E}\Big[p\Big(\frac{Z_0^-}{Z_0^+}\Big)\Big] = p\Big(\frac{z^-}{z^+}\Big).$$

Particular case: the Cardy's crossing probability formula

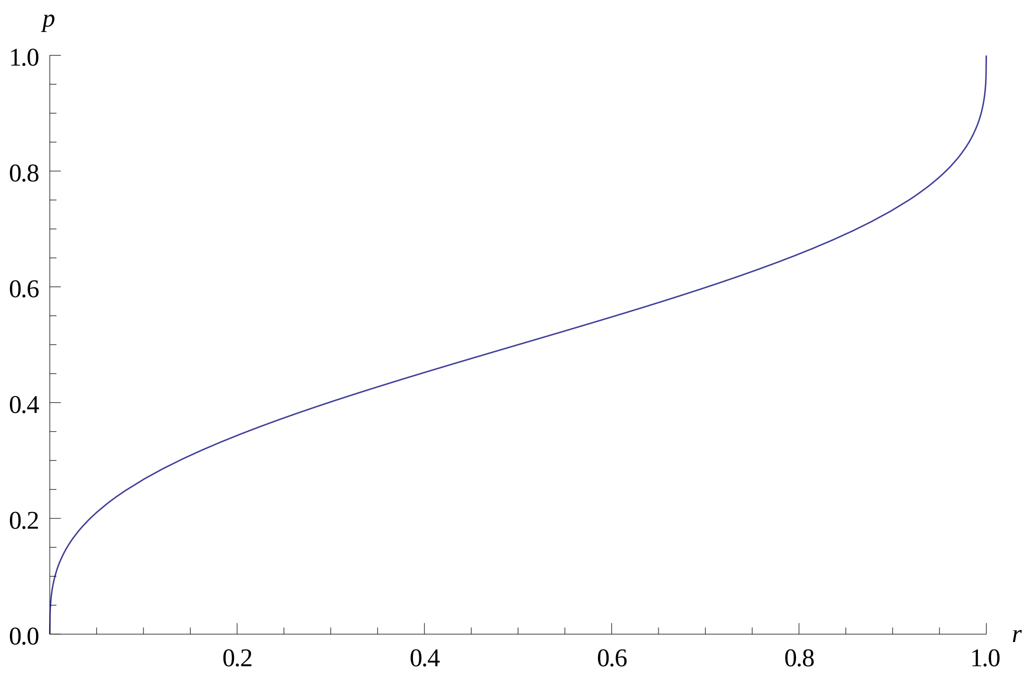


Figure 3.1: A plot of the boundary touching probability at $\kappa = 6$.

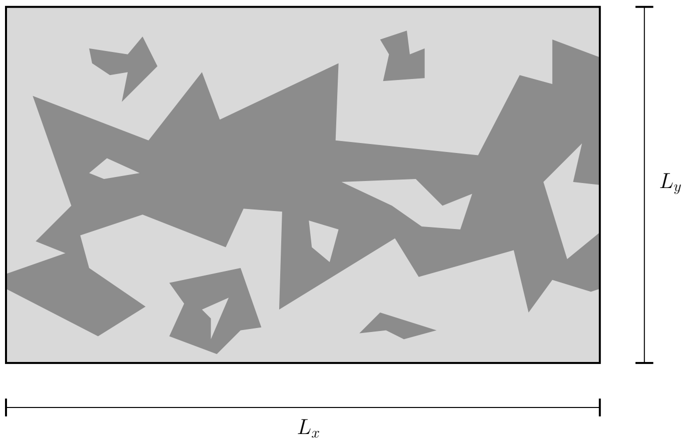


Figure 3.2: In percolation theory one is interested in the box crossing events, that is, existence of a connected open cluster touching the left and right sides of an $L_x \times L_y$ rectangle.

The dimension of the SLE trace

As a second application of the same technique, let us consider the determination of the fractal dimension of the SLE curve. Below, γ denotes a chordal SLE$_\kappa$ trace.

The question of dimension of the curve is intimately related to the probability that the curve gets close to a given point. For $z \in \mathbb{C}$ and $\varepsilon > 0$ denote by $B_\varepsilon(z) = \{w \in \mathbb{C} \, : \, |w - z| < \varepsilon\}$ the ball of radius ε centered at z. We will argue that

$$\mathsf{P}\Big[\gamma \cap B_\varepsilon(z) \neq \emptyset\Big] \;\sim\; \varepsilon^\alpha, \tag{3.1}$$

with some $\alpha > 0$ which depends on κ, but which is the same for all $z \in \Lambda$. The number α gives us the dimension of the curve at least in the following sense. If we cover an open bounded $A \subset \Lambda$ by $n_\varepsilon \sim \varepsilon^{-2}$ balls of radius ε, with centers $z_1, z_2, \ldots, z_n$, then the expected number of these balls that the curve visits is

$$\mathsf{E}\Big[\#\{j = 1, 2, \ldots, n_\varepsilon \, : \, \gamma \cap B_\varepsilon(z_j) \neq \emptyset\}\Big] \;=\; \sum_{j=1}^{n_\varepsilon} \mathsf{P}\Big[\gamma \cap B_\varepsilon(z_j) \neq \emptyset\Big] \;\sim\; n_\varepsilon \, \varepsilon^\alpha \sim \varepsilon^{\alpha-2}.$$

If the fractal dimension of the curve is d, then the typical number of ε-balls in A that the curve visits should be of order ε^{-d}, and in this sense the fractal dimension is

$$d \;=\; 2 - \alpha.$$

In fact an estimate of the type of Equation (3.1) easily leads to an (almost sure) upper bound $\dim_{\mathrm{Hausdorff}}(\gamma) \leq 2 - \alpha$ on the Hausdorff dimension of γ. To obtain the corresponding (almost sure) lower bound is much more involved — this was originally achieved by Beffara in [Bef08], and the interested reader will find a relatively clear proof in the lecture notes of Lawler [Law10].

We will give a heuristic derivation of the asymptotics of the probability to visit a small ball, in the spirit of Equation (3.1). It would not be very difficult, with a minor modification of the definitions, to fill in the gaps in the argument and to make it rigorous. However, the technicalities involved would take us a bit too far from the main point, so here we content ourselves with the idea and refer the interested reader to [RS05, Bef08, Law10] for the details.

We denote by $\gamma^{(\Lambda;a,b)}$ the chordal SLE$_\kappa$ trace in domain $(\Lambda; a, b)$, and we again mostly work in our reference domain $(\mathbb{H}; 0, \infty)$. Let us assume that there exists and $\alpha > 0$ such that the limit

$$\lim_{\varepsilon \searrow 0} \; \varepsilon^{-\alpha} \; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_\varepsilon(z) \neq \emptyset\Big] \;=\; G(z)$$

exists and is positive for all $z \in \mathbb{H}$. Such a function $G(z)$, giving the z dependence of the constant in front of the asymptotic visiting probability of small balls, is sometimes called the chordal SLE$_\kappa$

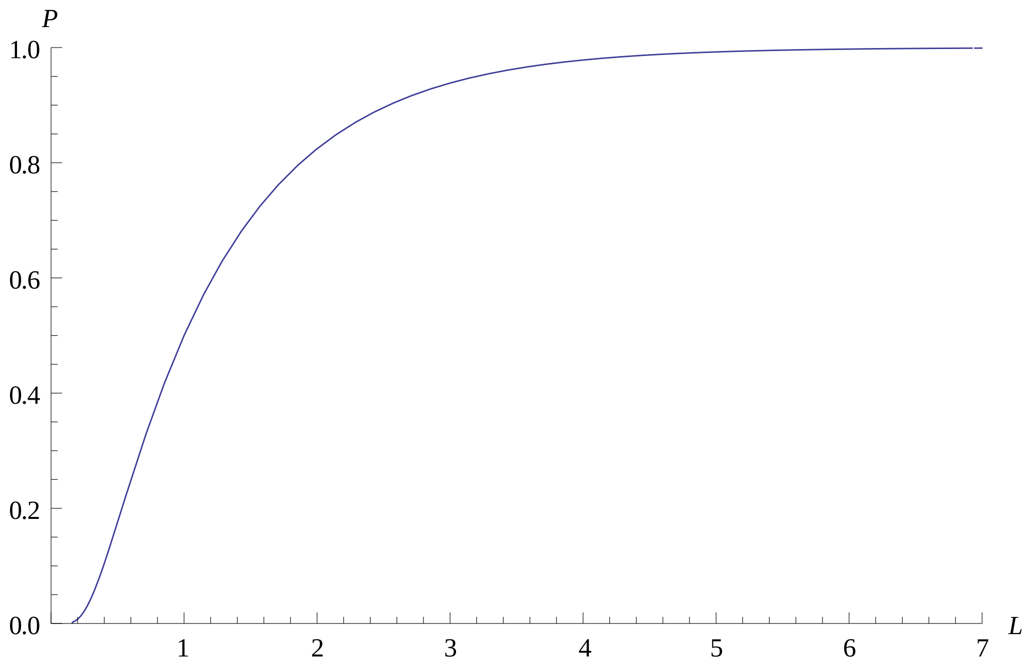


Figure 3.3: For an SLE_6 curve started from a corner of an $L \times 1$ rectangle, the probability to touch one of the opposite sides of the rectangle before the other, is the scaling limit of percolation crossing probabilities.

Green's function in the domain $(\mathbb{H}; 0, \infty)$. We immediately note that by the scale invariance, Proposition 1, for any $\lambda > 0$ we have

$$\mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_\varepsilon(z) \neq \emptyset\Big] \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_{\lambda\varepsilon}(\lambda z) \neq \emptyset\Big],$$

so after multiplying by $\varepsilon^{-\alpha}$ and taking the limit we see that

$$G(z) \;=\; \lambda^\alpha \, G(\lambda z).$$

Thus we can write

$$G(x + \mathrm{i}y) \;=\; y^{-\alpha} \, H\Big(\frac{x}{y}\Big).$$

Let us then construct the martingales appropriate for the problem. By definition, the conditional probabilities

$$\mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_\varepsilon(z) \neq \emptyset \,\Big|\, \gamma^{(\mathbb{H};0,\infty)}[0,t]\Big]$$

form a martingale, so let us define, for $\varepsilon > 0$ the martingales

$$M_t^{(\varepsilon)} \;=\; \varepsilon^{-\alpha} \, \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_\varepsilon(z) \neq \emptyset \,\Big|\, \gamma^{(\mathbb{H};0,\infty)}[0,t]\Big]$$

which take into account the asymptotics of the probabilities. By the domain Markov property, conditionally on the initial segment $\gamma^{(\mathbb{H};0,\infty)}[0,t]$, the law of $\gamma^{(\mathbb{H};0,\infty)}[t,\infty)$ is that of $\gamma^{(H_t;\gamma(t),\infty)}$, so we have

$$M_t^{(\varepsilon)} \;=\; \varepsilon^{-\alpha} \, \mathsf{P}\Big[\gamma^{(H_t;\gamma(t),\infty)} \cap B_\varepsilon(z) \neq \emptyset\Big].$$

Furthermore, using the conformal mapping $z \mapsto g_t(z) - X_t$ from $(H_t; \gamma(t), \infty)$ to $(\mathbb{H}; 0, \infty)$, and conformal invariance of chordal SLE, we can write

$$\begin{aligned} M_t^{(\varepsilon)} &= \varepsilon^{-\alpha} \, \mathsf{P}\Big[\Big(g_t(\gamma^{(H_t;\gamma(t),\infty)}) - X_t\Big) \cap \Big(g_t(B_\varepsilon(z) - X_t\Big) \neq \emptyset\Big] \\ &= \varepsilon^{-\alpha} \, \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap \Big(g_t(B_\varepsilon(z) - X_t\Big) \neq \emptyset\Big]. \end{aligned}$$

Observe that the image of a small ball $B_\varepsilon(z)$ under a conformal map is approximately a small ball at the image point, with radius multiplied by the absolute value of the derivative of the map. Therefore we may write an approximation

$$M_t^{(\varepsilon)} \approx \varepsilon^{-\alpha} \, \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_{|g_t'(z)|\,\varepsilon}\big(g_t(z) - X_t\big) \neq \emptyset\Big],$$

which would in fact be an equality had we used an appropriate conformally invariant notion of radius. Considering the limit $\varepsilon \searrow 0$ of the martingales, $M_t = \lim_\varepsilon M_t^{(\varepsilon)}$, the approximation above says that the process

$$M_t = |g_t'(z)|^\alpha \, G\big(g_t(z) - X_t\big)$$

should be a local martigale. As before, we can now write the Itô derivative and obtain a differential equation for G. Indeed, first note that $Z_t = g_t(z) - X_t$ satisfies the (complex) Bessel type equation

$$\mathrm{d}Z_t = \frac{2}{Z_t}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_t,$$

or in terms of the real and imaginary parts $Z_t = R_t + \mathrm{i}\, I_t$

$$\mathrm{d}R_t = \frac{2R_t}{R_t^2 + I_t^2}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_t, \qquad \mathrm{d}I_t = \frac{-2I_t}{R_t^2 + I_t^2}\,\mathrm{d}t.$$

We also need the time derivative of $|g_t'(z)|$, which is easiest to get in the logarithmic form: one reckognizes the derivative of the Loewner equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\big(g_t'(z)\big) = \frac{-2}{(g_t(z) - X_t)^2},$$

and then one takes the real part

$$\mathrm{d}\Big(\log|g_t'(z)|\Big) = \frac{-2(R_t^2 - I_t^2)}{(R_t^2 + I_t^2)^2}\,\mathrm{d}t.$$

Taking into account the scale invariance $G(x + \mathrm{i}y) = y^{-\alpha} H(x/y)$ our local martingale reads

$$M_t = \Big(\frac{|g_t'(z)|}{I_t}\Big)^\alpha H\Big(\frac{R_t}{I_t}\Big).$$

Applying Itô's formula we get

$$\mathrm{d}M_t = \frac{8\,|g_t'(z)|^\alpha}{I_t^{2+\alpha}}\Big(1 + r_t^2\Big)^{-2}\Big\{\alpha\, H(r_t) + (1 + r_t^2)\, r_t\, H'(r_t) + \frac{\kappa}{8}(1 + r_t^2)^2\, H''(r_t)\Big\}\,\mathrm{d}t + \Big(\cdots\Big)\,\mathrm{d}B_t,$$

where we have denoted the ratio of real and imaginary parts by $r_t = \frac{R_t}{I_t}$. Because the process is a local martingale, the $\mathrm{d}t$ term has to vanish and therefore we require

$$\alpha\, H(r) + (1 + r^2)\, r\, H'(r) + \frac{\kappa}{8}(1 + r^2)^2\, H''(r) = 0.$$

Solutions to this differential equation exist for any α, and correspondingly there are local martingales of the desired form. However, the solution we are interested in should be positive. Once we observe that the equation is an eigenvalue equation of Sturm-Liouville type, the eigenvalue being $-\alpha$, and we recall that a positive solution must have the unique extremal eigenvalue, the number α can be determined. We guess that the positive solution is of the form

$$H(r) = (1 + r^2)^\beta,$$

and then solve for β and α

$$\beta = \frac{\kappa - 8}{2\kappa}, \qquad \alpha = 1 - \frac{\kappa}{8}.$$

The result makes sense when $\alpha \geq 0$, that is, for $\kappa \leq 8$.

We in particular get the fractal dimension of the chordal SLE_κ trace stated in Theorem **??**

$$d = 2 - \alpha = 1 + \frac{\kappa}{8},$$

and we have even found how the probability to visit small balls depends on the position of the ball

$$\begin{aligned} \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap B_\varepsilon(x + \mathrm{i}\, y) \neq \emptyset\Big] &\sim \Big(\frac{\varepsilon}{y}\Big)^{1-\frac{\kappa}{8}} \frac{1}{(1 + \frac{x^2}{y^2})^{\frac{8-\kappa}{2\kappa}}} \\ &= \Big(\frac{\varepsilon}{y}\Big)^{1-\frac{\kappa}{8}} \Big(\sin(\arg(x + \mathrm{i}\, y))\Big)^{\frac{8-\kappa}{\kappa}}. \end{aligned} \tag{3.2}$$

Not surprisingly, for $\kappa < 8$, among balls whose center is at a given height y, the visiting probability gets smaller when the center of the ball gets further away from the starting point 0 of the SLE curve. The limiting case $\kappa = 8$ suggests a constant probability to visit any ball, and indeed a more careful analysis would show that the SLE trace then passes through any point almost surely.

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