Scaling limits of random fields with long range dependence[1]

Lasse Leskelä

Centrum voor Wiskunde en Informatica

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[1] Joint work with I. Kaj, I. Norros, V. Schmidt

Introduction

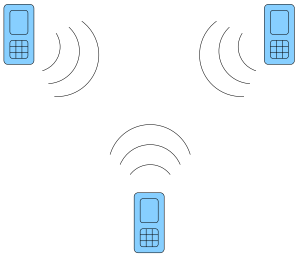
Outline

- Motivation
- Poisson noise model
- High-density limits
- Proofs

Earlier work:

- Taqqu and Levy (1986)
- Mikosch, Resnick, Rootzén, and Stegeman (2002)

Spatial noise generated by independent radio sources



- (X_j, R_j) location and transmission range of source j
- C shape of the transmission area
- λ mean density of the sources
- ρ mean transmission range

Describe the aggregate noise field

$$J_{\lambda,\rho}(x) = \#\{j : x \in X_j + R_j C\}$$

as $\lambda \to \infty$ and $\rho \to 0$.

Poisson grain field

Random scalar field on $\mathbb{R}^d$

$$J_{\lambda,\rho}(x) = \sum_{j=1}^{\infty} 1(x \in G_j),$$

generated by the *grains*

$$G_j = X_j + (\rho V_j)^{1/d} C$$

- deterministic shape: $C \subset \mathbb{R}^d$ bounded with unit volume
- random locations: $X_j \in \mathbb{R}^d$ independently scattered
- random volumes: $V_j \geq 0$ i.i.d. with unit mean
- model parameters: mean density λ and mean volume ρ

Poisson grain field in dimension one

For $C = [0, 1)$ in dimension one,

$$J_{\lambda,\rho}(t) = \sum_{j=1}^{\infty} 1(t \in X_j + (\rho V_j)^{1/d} C)$$
$$= \sum_{j=1}^{\infty} 1(X_j \leq t < X_j + \rho V_j)$$

This is the $M/G/\infty$ queueing process:

- arrival instants X_j Poisson with intensity λ
- service times ρV_j i.i.d. with mean ρ

Heavy tails and regular variation

Assume

- $\mathrm{P}(V_j > v) = L(v)v^{-\gamma}$
- $L(v)$ *slowly varying*:

$$\lim_{v\to\infty} \frac{L(\theta v)}{L(v)} = 1, \quad \text{for all } \theta > 0$$

Heavy tails when $\gamma \in (1,2)$:

- $\mathrm{E}\, V_j < \infty$
- $\mathrm{E}\, V_j^2 = \infty$

Examples

- slowly varying: any $f(v)$ with $\lim_{v\to\infty} f(v) > 0$
- not slowly varying: $f(v) = e^{-v}$, $f(v) = 2 + \sin(v)$

High-intensity $M/G/\infty$ process with heavy tails: (I) Long service times

Assume $P(V_j > v) = L(v)v^{-\gamma}$ with $\gamma \in (1, 2)$

Theorem (Mikosch, Resnick, Rootzén, Stegeman, 2002)
Assume $\lambda \to \infty$ and $\rho \to 0$ such that $\lambda \rho^\gamma L(1/\rho) \to \infty$. Then

$$\int_0^t \left(\frac{J_{\lambda,\rho}(s) - \mathrm{E}\, J_{\lambda,\rho}(s)}{c_{\lambda,\rho}} \right) ds \xrightarrow{fd} B_H(t).$$

B_H is *fractional Brownian motion* with $H = (3 - \gamma)/2$:

- Gaussian process
- $\mathrm{E}\, B_H(t) = 0$
- $\mathrm{E}\, B_H(s) B_H(t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$

High-intensity $M/G/\infty$ process with heavy tails: (II) Short service times

Theorem (Mikosch et al., 2002)

Assume $\lambda \to \infty$ *and* $\rho \to 0$ *such that* $\lambda \rho^\gamma L(1/\rho) \to 0$. *Then*

$$\int_0^t \left(\frac{J_{\lambda,\rho}(s) - \mathrm{E}\, J_{\lambda,\rho}(s)}{c_{\lambda,\rho}} \right) ds \xrightarrow{fd} \Lambda_\gamma(t).$$

Λ_γ is γ-*stable Lévy process* with unit skewness:

- independent increments
- γ-stable marginals with

$$\mathrm{E}\, e^{i\theta\Lambda_\gamma(t)} = e^{-t|\theta|^\gamma(1 - i\,\mathrm{sgn}(\theta)\tan(\pi\gamma/2))}$$

Poisson grain fields in higher dimensions

For dimensions $d > 1$,

$$\lim_{\lambda\to\infty,\rho\to 0} J_{\lambda,\rho}(x) = ?$$

Problems:

- The limit functions do not have point values (cf. derivative of Brownian motion)
- The cumulative process $\int_0^r J_{\lambda,\rho}(s\theta)\,ds$ does not characterize spatial correlation

Solution:

- Consider the *random linear functional*

$$J_{\lambda,\rho}(\phi) = \int_{\mathbb{R}^d} J_{\lambda,\rho}(x)\phi(x)\,dx$$

- For example for $d = 1$ with $\phi = 1_{[0,t]}$,

$$J_{\lambda,\rho}(\phi) = \int_0^t J_{\lambda,\rho}(s)\,ds$$

Poisson integral representation of $J_{\lambda,\rho}(x)$

$$J_{\lambda,\rho}(x) = \sum_{j=1}^{\infty} 1(x \in X_j + (\rho V_j)^{1/d} C)$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} 1(x \in y + v^{1/d} C)\, N_{\lambda,\rho}(dy, dv),$$

- $N_{\lambda,\rho}$ is a Poisson random measure on with intensity

$$\mathrm{E}\, N_{\lambda,\rho}(\phi) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \phi(x, v)\, dx\, \mathrm{P}(\rho V \in dv)$$

Poisson integral representation of $J_{\lambda,\rho}(\phi)$

- $J_{\lambda,\rho}$ as random linear functional

$$J_{\lambda,\rho}(\phi) = \int_{\mathbb{R}^d} J_{\lambda,\rho}(x)\,\phi(x)\,dx$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} v\, m_\phi(x,v)\, N_{\lambda,\rho}(dx,dv)$$

- m_ϕ are the averages

$$m_\phi(x,v) = \frac{\int_{v^{1/d}C} \phi(x+y)\,dy}{\int_{v^{1/d}C} dy}$$

High-density limit for light-tailed volumes

Theorem
Assume $\mathrm{E}\, V^2 < \infty$. *Then, as* $\lambda \to \infty$ *and* $\rho \to 0$,

$$\frac{J_{\lambda,\rho}(\phi) - \mathrm{E}\, J_{\lambda,\rho}(\phi)}{\rho(\lambda\, \mathrm{E}\, V^2)^{1/2}} \xrightarrow{d} W(\phi), \quad \phi \in L^1 \cap L^2.$$

W is the *white Gaussian noise* on $\mathbb{R}^d$:

- Gaussian random linear functional
- $\mathrm{E}\, W(\phi) = 0$
- $\mathrm{E}\, W(\phi) W(\psi) = \int_{\mathbb{R}^d} \phi(x)\psi(x)\, dx$

Note that for $d = 1$:

- $W(1_{[0,t]}) = \int_0^t W(ds) = B(t)$
- $B(t)$ is Brownian motion

Grains with heavy-tailed volumes

- Assume $\mathrm{P}(V > v) = v^{-\gamma} L(v)$ for $\gamma \in (1, 2)$
- Then for small ρ,

$$\mathrm{E} \sum_{j=1}^{\infty} 1(x \in G_j) 1(|G_j| > 1) = \lambda\rho \int_{1/\rho}^{\infty} v \, \mathrm{P}(V \in dv)$$

$$\sim \frac{\lambda\rho^{\gamma} L(1/\rho)}{1 - \gamma^{-1}}$$

- Three asymptotical regimes:

Small-grain scaling	$\lambda \, \rho^{\gamma} L(1/\rho) \to 0$
Intermediate scaling	$\lambda \, \rho^{\gamma} L(1/\rho) \to \gamma^{-1}$
Large-grain scaling	$\lambda \, \rho^{\gamma} L(1/\rho) \to \infty$

Three heavy-tailed limits: (I) Small-grain scaling

Theorem

If $\lambda \to \infty$ and $\rho \to 0$ such that $\lambda \rho^\gamma L(1/\rho) \to 0$, then

$$\frac{J_{\lambda,\rho}(\phi) - \mathrm{E}\, J_{\lambda,\rho}(\phi)}{\rho (1/(1-F))^{\leftarrow}(\gamma\lambda)} \xrightarrow{d} \Lambda_\gamma(\phi), \quad \phi \in L^1 \cap L^2.$$

Λ_γ is the independently scattered γ-stable random linear functional on $\mathbb{R}^d$ characterized by

$$\mathrm{E}\, e^{i\Lambda_\gamma(\phi)} = e^{-\sigma_\phi^\gamma \left(1 - i\beta_\phi \tan(\frac{\pi\gamma}{2})\right)},$$

where $\sigma_\phi = ||\phi||_\gamma$, $\beta_\phi = ||\phi||_\gamma^{-\gamma}(||\phi_+||_\gamma^\gamma - ||\phi_-||_\gamma^\gamma)$, and

$$||\phi||_\gamma = c_{\gamma,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\phi(x)\phi(y)}{|x-y|^{(1-\gamma)d}}\, dx\, dy$$

Three heavy-tailed limits: (II) Intermediate scaling

Theorem

If $\lambda \to \infty$ and $\rho \to 0$ such that $\lambda \rho^\gamma L(1/\rho) \to \gamma^{-1}$, then

$$J_{\lambda,\rho}(\phi) - \mathsf{E}\, J_{\lambda,\rho}(\phi) \xrightarrow{d} J^*_{\gamma,C}(\phi), \quad \phi \in L^1 \cap L^2.$$

$J^*_{\gamma,C}$ is the random linear functional on $L^1 \cap L^2$ with

$$J^*_{\gamma,C}(\phi) = \int_{\mathbb{R}^d} \int_0^\infty v m_\phi(x,v) \left(N_\gamma(dx,dv) - dx\, v^{-\gamma-1} dv \right),$$

where $N_\gamma(dx,dv)$ is a PRM on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $dx\, v^{-\gamma-1} dv$

Three heavy-tailed limits: (III) Large-grain scaling

Theorem
If $\lambda \to \infty$ and $\rho \to 0$ such that $\lambda \rho^\gamma L(1/\rho) \to \infty$, then

$$\frac{J_{\lambda,\rho}(\phi) - \mathrm{E}\, J_{\lambda,\rho}(\phi)}{(\gamma \lambda \rho^\gamma L(1/\rho))^{1/2}} \xrightarrow{d} W_{\gamma,C}(\phi), \quad \phi \in L^1 \cap L^2.$$

$W_{\gamma,C}$ is the Gaussian random linear functional on $L^1 \cap L^2$

- $\mathrm{E}\, W_{\gamma,C}(\phi) = 0$
- $\mathrm{E}\, W_{\gamma,C}(\phi) W_{\gamma,C}(\psi) = \iint \phi(x) K_{\gamma,C}(x-y) \phi(y)\, dx\, dy$, where

$$K_{\gamma,C}(x) = \int_0^\infty \left| v^{-1/d} x + C \cap C \right| v^{-\gamma} dv$$

Properties of the limits

Self-similarity

- A random linear functional J is *self-similar*, if

$$J(\phi \circ a^{-1}) \stackrel{d}{=} a^{Hd} J(\phi), \quad \text{for all } \phi \text{ and } a > 0$$

- The large-grain limit $W_{\gamma,C}$ is self-similar with $H = (3 - \gamma)/2$
- The small-grain limit Λ_γ is self-similar with $H = 1/\gamma$

Stationarity

- All limits $W_{\gamma,C}, J^*_{\gamma,C}, \Lambda_\gamma$ are stationary

Correlations

- $W_{\gamma,C}$ and $J^*_{\gamma,C}$ share the same second order statistics
- Λ_γ has infinite second moments

Fractional Gaussian noise

When the shape C is the closed unit ball,

$$W_{\gamma,C} \stackrel{d}{=} cW_H, \quad H = (3-\gamma)/2,$$

where W_H is *fractional Gaussian noise* on $L^1 \cap L^2$:

- $\mathrm{E}\, W_H(\phi) = 0$
- $\mathrm{E}\, W_H(\phi) W_H(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\phi(x)\psi(y)}{|x-y|^{(2-2H)d}}\, dx\, dy$

The covariance kernel is called the *Riesz potential*

Fractional Gaussian noise in dimension $d = 1$

In dimension one,

$$W_H(1_{[0,t]}) = \int_0^t W_H(ds) = c_H B_H(t),$$

where B_H is fractional Brownian motion.

Hence

$$W_H = c_H \frac{d}{dt} B_H$$

in the sense of generalized functions.

White noise representation of W_H

For $H \in (1/2, 1)$,

$$W_H(\phi) \stackrel{d}{=} c_{H,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\phi(y)dy}{|x-y|^{(3/2-H)d}} W(dx),$$

where W is white Gaussian noise on L^2

Proofs of Theorems 1–4

The proofs are based on

(i) Fourier transform of the Poisson noise

(ii) Extended Potter's bounds

(iii) Hardy–Littlewood maximal function theory

Fourier transform of the Poisson noise

When N is a PRM with intensity η and

$$\int (|\phi| \wedge \phi^2) d\eta < \infty,$$

then the stochastic integral $\int \phi(dN - d\eta)$ exists, and

$$\mathsf{E}\, e^{i \int \phi(dN - d\eta)} = e^{\int \Psi(\phi) d\eta}$$

with

$$\Psi(v) = e^{iv} - 1 - iv.$$

Note: linearity $\implies$ no need to consider $\sum_{j=1}^{n} \theta_j \phi_j$

Potter's bounds

Lemma

Let $V \geq 0$ *be a random variable such that* $P(V > v) = L(v)v^{-\gamma}$ *for some* $\gamma > 0$*. Then for any* $\epsilon \in (0, \gamma)$ *there exist positive numbers* c_ϵ *and* ρ_ϵ *such that for all* $\rho \in (0, \rho_\epsilon)$*,*

$$\frac{L(v/\rho)}{L(1/\rho)} \leq c_\epsilon \left(v^{-\epsilon} \vee v^{+\epsilon} \right) \quad \forall v > 0.$$

Hardy–Littlewood maximal theorem

Let ϕ_* be the *Hardy–Littlewood maximal function* of ϕ,

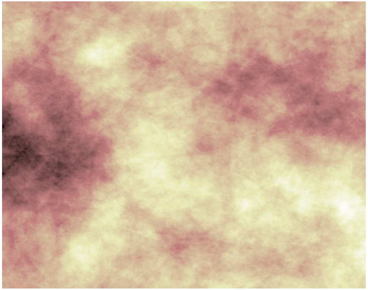
$$\phi_*(x) = \sup_{v>0} \frac{\int_{v^{1/d}C} |\phi(x+y)|\, dy}{\int_{v^{1/d}C}\, dy}$$

For all $p > 1$,

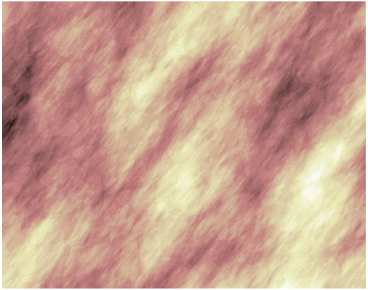
$$\phi \in L^p \implies \phi_* \in L^p$$

Illustration

$$J_{\lambda,\rho}(\phi) \sim \mathrm{E}\, J_{\lambda,\rho}(\phi) + (\gamma\lambda\rho^{\gamma})^{1/2}\, W_H(\phi)$$



Symmetric grains



Asymmetric grains

References

- M. Taqqu, J. B. Levy.
  Using renewal processes to generate long-range dependence and high variability.
  In *Dependence in Probability and Statistics*, pp. 73–89, Birkhäuser, 1986.
- T. Mikosch, S. Resnick, H. Rootzén, A. Stegeman.
  Is network traffic approximated by stable Lévy motion or fractional Brownian motion?
  Ann. Appl. Probab. **12**:23–68, 2002.
- I. Kaj, I. Norros, L. Leskelä, V. Schmidt.
  Scaling limits of random fields with long-range dependence.
  Ann. Probab. To appear, 2007.