Moment-based parameter estimation in binomial random intersection graph models

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Abstract

Random intersection graphs (RIG) can be used as parsimonious models of large and sparse networks. We derive moment-based parameter estimators for a class of RIG models and prove their consistency when only a subset of the data is used for estimation.

Random intersection graphs

RIGs are models of undirected and unweighted graphs, where a link is present between two nodes exactly when they share a common attribute (e.g., a hobby or an interest). The model $G(n, m_n, p_n)$ is specified as follows:

- n, the number of nodes
- m_n, the number of attributes
- $p_n \in (0,1)$, the probability that node i has attribute k
- $V_i \subset \{1,2,\dots,m_n\}$, the (random) set of attributes assigned to node i
- Node i links to node j if and only if $|V_i \cap V_j| \neq 0$.
- The attributes are assigned independently, so that $|V_i| \sim Bin(m_n, p_n)$.

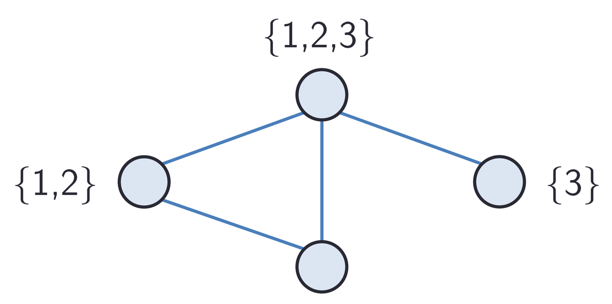


Figure 1: A realization of a random intersection graph with $n = 4$.

We consider a sequence of graphs $G(n, m_n, p_n)$ and its limiting behavior as $n \longrightarrow \infty$. In the limit, we wish to have

- a nontrivial average degree of the nodes, and
- a nontrivial *clustering coefficient* $\mathbb{P}(j \leftrightarrow k \mid i \leftrightarrow j, i \leftrightarrow k)$.

These are achieved when

$$p_n = \frac{\lambda}{\mu} n^{-1} \quad \text{and} \quad m_n = \frac{\mu^2}{\lambda} n, \qquad (1)$$

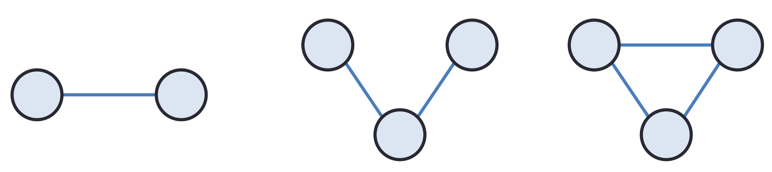
where

- λ is the limiting expected degree of a node, and
- μ is the limiting expected number of attributes of a node.

Question: How can we estimate λ and μ from a single observed network?

Asymptotic subgraph counts

Parameter estimates can be based on counting the numbers of certain subgraphs in the observed network. Consider the following subgraphs:



K_2 S_2 K_3

Let N_{K_2}, N_{S_2} and N_{K_3} be the empirical counts of links, 2-stars and triangles. Under model (1) it holds that

$$\mathbb{E}[N_{K_2}] \sim n^2 \mu^2 m_n^{-1},$$
$$\mathbb{E}[N_{S_2}] \sim n^3 (1+\mu) \mu^3 m_n^{-2},$$
$$\mathbb{E}[N_{K_3}] \sim n^3 \mu^3 m_n^{-2}.$$

Parameter estimators are found by solving for λ and μ and replacing $\mathbb{E}[N_*]$ with N_*. The variances of N_* can be bounded by using the following lemma.

Lemma 1 *The probability that a random intersection graph $G(|G|, m_n, p_n)$ contains a connected subgraph S with $|S|$ nodes satisfies*

$$\mathbb{P}(S \subset G) = O\left(|G|^{|S|} m_n p_n^{|S|}\right).$$

Induced subgraph sampling and consistency

Counting the triangles in the graph with a naïve method requires $O(n^3)$ operations. One may reduce the computation time by only using an *induced subgraph* $G^{(n_0)}$ of the data, i.e., a subset of $n_0 < n$ nodes and the links between them.

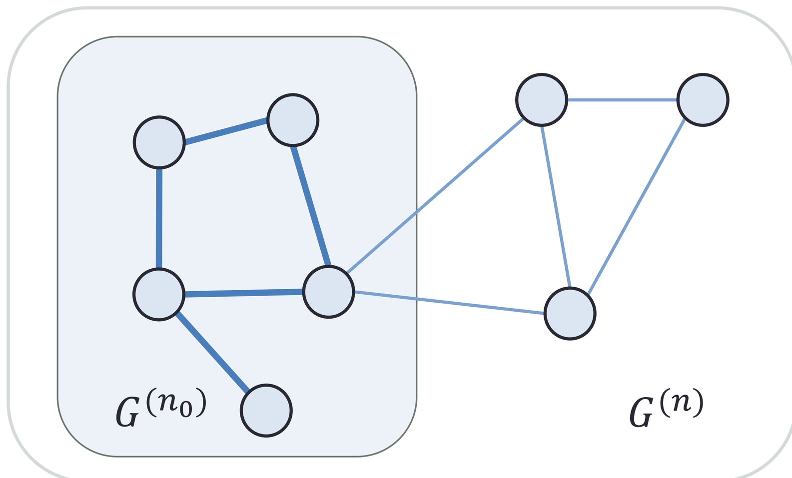


Figure 2: A data set $G^{(n)}$ with $n = 8$ and an induced subgraph $G^{(n_0)}$ with $n_0 = 5$.

Using the asymptotic subgraph counts we obtain the following estimators for λ and μ.

$$\hat{\lambda} = \frac{n}{n_0^2} \sum_{i \in G^{(n_0)}} \deg_{G^{(n_0)}}(i)$$

$$\hat{\mu}_1 = \frac{N_{S_2}(G^{(n_0)})}{3N_{K_3}(G^{(n_0)})} - 1$$

$$\hat{\mu}_2 = \left(\frac{n_0 N_{S_2}(G^{(n_0)})}{2N_{K_2}(G^{(n_0)})^2} - 1\right)^{-1}$$

Question: Given a sufficiently large n and n_0, are these estimators close to the true parameter values?

The following theorems confirm that this is the case, in a suitable sense:

Theorem 1 *Estimator $\hat{\lambda}$ is consistent, i.e., $\hat{\lambda} \xrightarrow{p} \lambda$, when $n_0 \gg n^{1/2}$. Moreover,*

$$\hat{\lambda}(G^{(n_0)}) = \lambda + O_p\left(\frac{n^{1/2}}{n_0}\right).$$

Theorem 2 *Estimators $\hat{\mu}_1$ and $\hat{\mu}_2$ are consistent when $n_0 \gg n^{2/3}$.*

The proofs are based on the second moment method, Lemma 1 and the continuous mapping theorem.

Simulations

- Parameters are estimated once for each $n = 50, 70, \dots, 1000$.
- Two sets of parameters, $(\lambda = 9, \mu = 3)$ and $(\lambda = 2, \mu = 0.5)$.
- The biases decrease rapidly as the size of the graph grows.
- $\hat{\mu}_1$ seems to be better than $\hat{\mu}_2$, but counting the triangles takes time.
- The data can be much larger ($n \approx 10^5$ with a simple implementation).

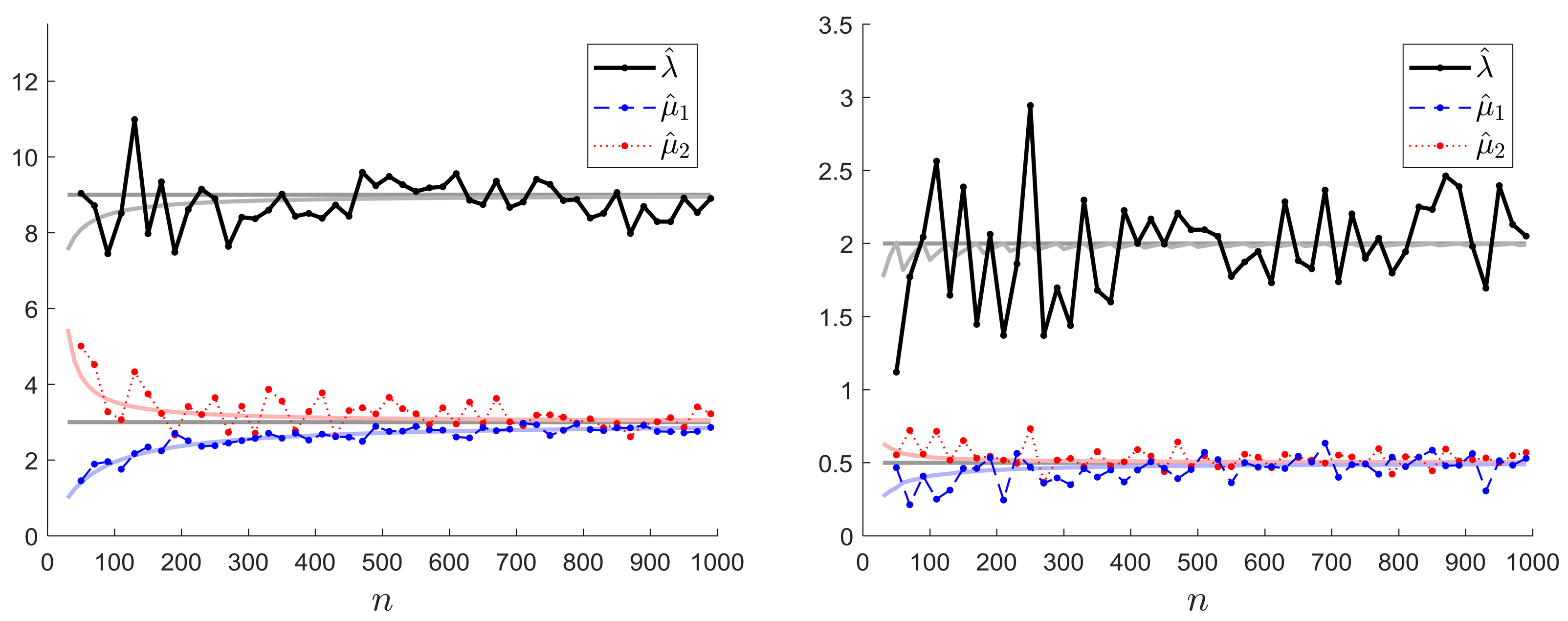


Figure 3: Estimated values for parameters (λ, μ) with $n_0 = n$ for simulated graphs of sizes $n = 50, \dots, 1000$.

References:

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