Exercise 4
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4.1 *Indicator random variable.* The *indicator* of an event $A \subset \Omega$ in a discrete probability space (Ω, P) is the $\{0,1\}$-valued random variable

$$1_A(\omega) = \begin{cases} 1, & \text{when } \omega \in A, \\ 0 & \text{else.} \end{cases}$$

(a) Find out the distribution of 1_A.

(b) Compute the expectation and variance of 1_A.

(c) Find out the distribution of $Z = 1_{A_1} + \cdots + 1_{A_n}$ when we assume that $1_{A_1}, 1_{A_2}, \ldots, 1_{A_n}$ are independent.

(d) Compute the expectation and variance of Z.

4.2 *Valtteri's roulette game.* Valtteri plays 300 rounds of roulette by betting one euro at each round to small numbers (1–18). Valtteri's initial capital equals $V_0 = 300$ euros. Denote the value of Valtteri's game account after t rounds byV_t.

(a) Let $U_1, \ldots, U_{300}$ be independent uniformly distributed random variables in $S = \{0, 1, \ldots, 36\}$. Define a function f such that the game account can be represented by,

$$V_t = V_0 + \sum_{s=1}^{t} f(U_s).$$

(b) What is the state space of the random variable V_{300}?

(c) Using part a) compute the expectation $\mathbb{E}V_{300}$.

(d) Let θ_s be the indicator of the event $\{U_s \in [1, 18]\}$. Explain why θ_s follows a $\mathrm{Ber}(p)$ distribution for some p and find out the value of p.

(e) Define a function g such that the game account can be represented as

$$V_t = V_0 + g\left(\sum_{s=1}^{t} \theta_s\right).$$

(f) Using part e) prove that

$$\mathbb{P}\left(\frac{V_{300} - V_0}{V_0} \geq 0.1\right) = \sum_{k=j}^{300} \binom{300}{k} p^k (1-p)^{300-k}$$

for some value of j and find out this value.

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4.3 *Nico's roulette game.* Nico plays 300 rounds of roulette by betting one euro at each round to the number 28. Nico's initial capital equals $V_0 = 300$ euros. Answer for Nico's part to the same questions as in Problem 4.2, when in part 4.2 d) the definition of θ_s has been updated to correspond to the indicator of the event $\{U_s = 28\}$.

4.4 *Gambler's ruin.* Heikki goes to a casino with purpose of increasing his initial capital to the level of n EUR in a game where each round yields +1 EUR with probability p and -1 EUR with probability $q = 1 - p$. Heikki's initial capital equals i EUR and he has decided to play unit he either reaches his target or loses his money. Let r_i be the probability that the game ends successfully for Heikki. During the lecture we found out that $r_0 = 0$, $r_n = 1$ and

$$r_i = pr_{i+1} + qr_{i-1}$$

for all $i = 1, \dots, n-1$. Solve r_i.

4.5 *Law of large number does not always hold.* Consider two independent collections of random variables such that

- $\{X_1, X_2, \dots\}$ are independent and identically distributed, and $\mathbb{E}(X_1) = 1$.
- $\{Y_1, Y_2, \dots\}$ are independent and identically distributed, and $\mathbb{E}(Y_1) = 2$.

Let us first flip a coin and then define

$$S_n = \begin{cases} X_1 + X_2 + \cdots + X_n, & \text{if we get heads,} \\ Y_1 + Y_2 + \cdots + Y_n, & \text{if we get tails.} \end{cases}$$

Prove that

(a) $\mathbb{E}(S_n/n) = 3/2$.

(b) $\mathbb{P}(|S_n/n - 3/2| > 1/4)$ does not converge to zero as n grows.

(c) Is the above observation in conflict with the law of large numbers?