5.1 *General properties of probability generating functions.* Let X be a $\mathbb{Z}_+$-valued random number and G_X its probability generating function.

(a) Explain why G_X is always defined on the set $[-1, 1]$.

(b) Prove that G_X maps the interval $[-1, 1]$ to itself.

(c) Compute $G_X(0)$ and $G_X(1)$.

(d) Can you say what $G_X(-1)$ tells about the distribution of X?

5.2 *Poisson distribution.* Let X be a Poisson-distributed random number with parameter $\lambda > 0$, so that

$$\mathbb{P}(X = k) \;=\; e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

(a) Compute the probability generating function G_X of X.

(b) For which values of t $G_X(t)$ is defined?

(c) Compute the expectation of X directly from the distribution of X.

(d) Compute the expectation of X using G_X.

(e) Which method was easier?

(f) Compute the variance of X.

5.3 *Number of grandchildren.* Mrs Galton gets N children during her lifetime, where

$$\mathbb{P}(N = k) = (1 - p)^k p, \quad k = 0, 1, 2, \dots$$

and $p \in (0, 1)$. (We say that N follows a geometric distribution on the set $\{0, 1, 2, \dots\}$.) Assume that each child, independently of the others, produces a random number of children which has the same distribution as N. Compute

(a) the probability generating function and expectation of the number of children,

(b) the probability generating function and expectation of the number of grandchildren,

(c) the probability that Mrs Galton gets at least two grandchildren.

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5.4 *Doubly stochastic Poisson distribution.* Consider a random number $N : \Omega \to \mathbb{Z}_+$, the distribution of which depends on an external random variable$L : \Omega \to \mathbb{Z}_+$ so that N has Poisson distribution with parameter ℓ when L equals ℓ. That is,

$$\mathbb{P}(N = k \,|\, L = \ell) \;=\; e^{-\ell}\frac{\ell^k}{k!}, \quad k \in \mathbb{Z}_+.$$

(a) Show that $\mathbb{P}(N = k) = \mathbb{E}e^{-L}\frac{L^k}{k!}$ for all $k \in \mathbb{Z}_+$.

(b) Compute $\mathbb{E}N$ when we assume that $\mathbb{E}L = \log 2$.

5.5 *Branching process with many initial individuals.* The branching process generated by one initial individuals is recursively defined by $Z_0 = 1$ and

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}, \quad n = 1, 2, \ldots,$$

where the random numbers $(X_{n,i})_{i\geq 1, n\geq 1}$ are independent and follow the same distribution as a generic $\mathbb{Z}_+$-valued random number X. During the lectures it was shown that the probability generating function of Z_n can be computed according to

$$G_{Z_n}(t) = G_X \circ \cdots \circ G_X(t), \quad n \geq 1,$$

where on the right we have the probability generating function of X iterated with itself n times.

(a) With the help of the above result, derive an expression for the probability generating function of Z_n in the generalized case where the initial population has three individuals who will produce offspring independently of each other, as in a population with one initial individuals.

(b) Can you generalize the result in part a) to the case where the initial population has a random number Y individuals?