University of Jyväskylä
Department of Mathematics and Statistics
MATS255 Markov Processes

Exercise 2
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2.1 *Simulation of an arbitrary random number.* Let X be a random real number with cumulative distribution function $F(x) = \mathbb{P}(X \le x)$. Define the the left-continuous inverse of F by

$$F^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : F(x) \ge y\}, \quad y \in (0,1).$$

(a) Show that $F^{\leftarrow}(y) \le x$ if and only if $y \le F(x)$.

(b) Show that $F^{\leftarrow}(U)$ is a random variable with the same distribution as X.

(c) Compute F and $F^{\leftarrow}$ for the random variable X which assumes value 1 with probability μ_1 and value 2 with probability $\mu_2 = 1 - \mu_1$.

2.2 *Update functions are not unique.* [Häg02, Ex. 3.2] Show that we get a valid update function for the Gothenburg weather transition matrix $P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$ by defining

$$\phi(s_1, x) = \begin{cases} s_1 & \text{for } x \in [0, 0.75), \\ s_2 & \text{for } x \in [0.75, 1], \end{cases} \qquad \phi(s_2, x) = \begin{cases} s_2 & \text{for } x \in [0, 0.75), \\ s_1 & \text{for } x \in [0.75, 1]. \end{cases}$$

Compare this with the update function in [Häg02, Example 3.1].

2.3 *Sojourn time in a state.* Consider a Markov chain (X_n) in $S = \{1, 2, \dots, k\}$ having a nonrandom initial state $i \in S$ and a transition matrix $P = (P_{i,j})_{i,j \in S}$ such that $P_{i,i} < 1$. Denote the sojourn time of (X_n) at state i by $T = \min\{n \ge 1 : X_n \ne i\}$.

(a) Compute the probability $\mathbb{P}(X_1 = i, X_2 = i)$.

(b) Compute the probability $\mathbb{P}(X_1 = i, X_2 = i, X_3 \ne i)$.

(c) Compute the probability $\mathbb{P}(T = n)$ for $n = 0, 1, 2, \dots$
Can you identify the probability distribution of T from this formula?

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2.4 *Partially observed Markov chain.* Let (X_n) be a Markov chain on $S = \{1, 2, \dots, k\}$ with initial distribution μ and transition matrix P. Define $Y_n = X_{rn}$, where r is a positive integer. Show that (Y_n) is a Markov chain with initial distribution μ and transition matrix P^r. (**Hint**: Recall Exercise 1.5 last week.)

2.5 *Markov chain on an infinite state space.* A Markov chain on a countably infinite state space S can be modeled with a transition matrix $(P_{i,j})_{i,j \in S}$ such that $P_{i,j} \geq 0$ for all i, j and $\sum_{j \in S} P_{i,j} = 1$ for all i. Given an a probability vector μ on S, define μP by

$$(\mu P)_j = \sum_{i \in S} \mu_i P_{i,j}, \quad j \in S.$$

(a) Show that the sum on right side above is finite for all j.

(b) Show that $\mu \mapsto \mu P$ maps probability vectors on S to probability vectors on S.

References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications.* Cambridge University Press, 2002.