University of Jyväskylä
Department of Mathematics and Statistics
MATS255 Markov Processes

Exercise 3
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3.1 *Transient state of a Markov chain.* [Häg02, Ex. 4.1] Consider the Markov chain $(X_0, X_1, \dots)$ with state space $S = \{s_1, s_2\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

(a) Draw the transition graph of this Markov chain.

(b) Show that the Markov chain is *not* irreducible (even though the transition matrix looks in some sense connected).

(c) What happens to X_n in the limit as $n \to \infty$?

3.2 *Aperiodicity condition for irreducible Markov chains.* [Häg02, Ex. 4.2] Show that if a Markov chain is irreducible and has a state s_i such that $P_{i,i} > 0$, then it is also aperiodic.

3.3 *Random chess moves.* [Häg02, Ex. 4.3]

(a) Consider a chessboard with a lone white king making random moves, meaning that at each move, he picks one of the possible squares to move to, uniformly at random. Is the corresponding Markov chain irreducible and/or aperiodic?

(b) Same question, but with the king replaced by a bishop.

(c) Same question, but instead with a knight.

3.4 *Oriented random walk on a torus.* [Häg02, Ex. 4.4] Let a and b be positive integers, and consider the Markov chain with state space

$$\{(x, y) : x \in \{0, \dots, a-1\}, y \in \{0, \dots, b-1\}\}$$

and the following transition mechanism: If the chain is in state (x, y) at time n, then at time $n+1$ it moves to $((x+1) \mod a, y)$ or $(x, (y+1) \mod b)$ with probability $\frac{1}{2}$ each.

(a) Show that this Markov chain is irreducible.

(b) Show that it is aperiodic if and only if $\gcd(a, b) = 1$.

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3.5 *Nonlattice integer sequences closed under addition.* Let A be a nonempty subset of $\mathbb{N} = \{1, 2, \dots\}$ which is closed under addition, so that $a, b \in A \implies a + b \in A$. Show that there exists an integer n_0 such that $n \in A$ for all $n \geq n_0$, under one of the following additional assumptions:

(a) $2 \in A$ and $3 \in A$.

(b) $3 \in A$ and $4 \in A$.

This exercise is a special instance of [Häg02, Lemma 4.1]

References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications.* Cambridge University Press, 2002.