4.1 *Coupon collector.* A company issues K different types of coupons. A collector desires a complete set. We suppose that each coupon he acquires is equally likely to be any of the K types, independently of the other acquisitions. Denote by X_n the number of distinct types of coupons the collector has after n acquisitions, and assume that the collector starts with no coupons.

(a) Show that (X_n) is a Markov chain on $S = \{0, \dots, K\}$, and draw its transition diagram.

(b) Denote by T_i the number of acquisitions required to gain i distinct types of coupons. What is the distribution of $\tau_i = T_{i+1} - T_i$ for $i = 1, \dots, K-1$?

(c) What is the mean number of acquisitions required to gather a complete set of coupons?

4.2 *Two independent Markov chains.* Let (X_n) be a MC with initial distribution $\mu^{(0)}$ and transition matrix P, and let (Y_n) a MC with initial distribution $\nu^{(0)}$ and transition matrix Q, in a finite state space S. Assume that (X_n) and (Y_n) are independent, and define $Z_n = (X_n, Y_n)$ for $n \geq 0$.

(a) Show that (Z_n) is a Markov chain in $S \times S$ having the transition matrix R with entries

$$R_{(i,j),(i',j')} = P_{i,i'} Q_{j,j'}, \quad (i,j) \in S \times S, \ (i',j') \in S \times S.$$

(b) What is the initial distribution of (Z_n)?

4.3 *Total variation distance.* [Häg02, Ex. 5.1] Let μ and ν be probability vectors on $S = \{1, 2, \dots, k\}$. Show that the total variation distance between μ and ν satisfies

$$\frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i| = \max_{B \subset S} |\mu(B) - \nu(B)|,$$

where $\mu(B) = \sum_{i \in B} \mu_i$. **Hint**: Consider the event $A_+ = \{i \in S : \mu_i \geq \nu_i\}$.

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4.4 *Markov chain convergence theorem fails for reducible MC's.* [Häg02, Ex. 5.2] Consider the reducible MC with transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

(a) Show that both $\pi = (0.375, 0.625, 0, 0)$ and $\pi' = (0, 0, 0.5, 0.5)$ are stationary distributions for this Markov chain.

(b) Find a probability vector $\mu^{(0)}$ such that the Markov chain with initial distribution $\mu^{(0)}$ satisfies $\mu^{(n)} \overset{\mathrm{TV}}{\to} \pi$.

(c) Find a probability vector $\mu^{(0)}$ such that the Markov chain with initial distribution $\mu^{(0)}$ satisfies $\mu^{(n)} \overset{\mathrm{TV}}{\to} \pi'$.

(d) Does the Markov chain convergence theorem [Häg02, Thm. 5.2] hold in this case?

4.5 *Markov chain convergence theorem fails for periodic MC's.* [Häg02, Ex. 5.3]. Consider the random knight on the empty chessboard as in Exercise 3.3. Assume that the knight starts at a nonrandom initial state, say `a1`. Show that the distribution $\mu^{(n)}$ of this MC does not converge in total variation.

References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications.* Cambridge University Press, 2002.