**5.1** *Markov chain with a symmetric transition matrix.* Consider a Markov chain $(X_n)$ on $S = \{1, \dots, k\}$ with a transition matrix $P$.

(a) Show that if $P$ is symmetric, then the uniform distribution on $S$ is reversible for $(X_n)$.

(b) A matrix is called doubly stochastic, if its entries are nonnegative and its row and column sums equal one. Show that if $P$ is doubly stochastic, then the uniform distribution on $S$ is a stationary distribution of $(X_n)$.

(c) If $P$ is doubly stochastic, does it follow that $(X_n)$ is reversible?

**5.2** *Random king on the chessboard.* [Häg02, Ex. 6.1]. Consider the random king on the empty chessboard as in Exercise 3.3. If you solved that problem correctly, then you know that the corresponding Markov chain is irreducible and aperiodic. By the Markov chain convergence theorem [Häg02, Thm 5.2], the chain therefore has a unique stationary distribution $\pi$. Compute $\pi$. **Hint:** This Markov chain is reversible.

**5.3** *Ehrenfest's urn.* [Häg02, Ex. 6.2]. Fix an integer $k$, and imagine two urns, each containing a number of balls, in such a way that the total number of balls in the two urns is $k$. At each integer time, we pick one ball at random (each with probability $\frac{1}{k}$) and move it to the other urn. If $X_n$ denotes the number of balls in the first urn, then $(X_0, X_1, \dots)$ forms a Markov chain with state space $\{0, \dots, k\}$

(a) Write down the transition matrix of this Markov chain.

(b) Show that the Markov chain is reversible with stationary distribution $\pi$ given by

$$\pi_i = \frac{k!}{i!(k-i)!} 2^{-k} \quad \text{for } i = 0, \dots, k.$$

(c) Show that the same distribution (known as the binomial distribution) also arises as the distribution of $\xi_1 + \cdots + \xi_k$, where $\xi_1, \dots, \xi_k$ are independent random variables uniformly distributed in $\{0, 1\}$.

(d) Can you give an intuitive explanation of why b) and c) give rise to the same distribution?

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**5.4** *Convergence of expectations of MC's.* Let $(X_n)$ be an irreducible aperiodic Markov chain on finite state space $S$. Denote by $\pi$ the stationary distribution of $(X_n)$. Prove that

$$\mathbb{E}f(X_n) \to \sum_i f(i)\pi_i \quad \text{as } n \to \infty$$

for any function $f : S \to \mathbb{R}$.

**5.5** *Time reversal.* [Häg02, Ex. 6.3]. Let $(X_0, X_1, \dots)$ be a reversible Markov chain with state space $S$, transition matrix $P$, and reversible distribution $\pi$. Show that if the chain is started with initial distribution $\mu^{(0)} = \pi$, then for any $n$ and any $i_0, i_1, \dots, i_n \in S$, we have

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P(X_0 = i_n, X_1 = i_{n-1}, \dots, X_n = i_0)$$

In other words, the chain is equally likely to make a tour through the states $i_0, \dots i_n$ in forwards as in backwards order.

# References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications.* Cambridge University Press, 2002.