University of Jyväskylä
Department of Mathematics and Statistics
MATS255 Markov Processes

Exercise 1
Mon 5 Nov 2012
L. Leskelä & M. Kuronen

1.1 *Random walk on a square.* [Häg02, E2.1] Consider a random walk on a square with vertices $\{v_1, v_2, v_3, v_4\}$, where the walker flips a fair coin and moves one step clockwise if he receives heads, and counterclockwise otherwise. This random walk can be modeled by the Markov chain with initial distribution $\mu_0 = (1, 0, 0, 0)$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

(a) Compute the square P^2 of the transition matrix P. How can we interpret P^2?

(b) Prove by induction that the distribution of the random walk after t time steps is given by

$$\mu_t = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}), & \text{for } t = 1, 3, 5, \ldots, \\ (\frac{1}{2}, 0, \frac{1}{2}, 0), & \text{for } t = 2, 4, 6, \ldots. \end{cases}$$

1.2 *Lazy random walk on a square.* [Häg02, E2.2] Suppose that we modify the random walk in Exercise 1.1 as follows. At each integer time, the random walker tosses *two fair coins.* The first coin is to decide whether to stay or go. If it comes up heads, he stays where he is, whereas if it comes up tails, he lets the second coin decide whether he should move one step clockwise, or one step counterclockwise. Write down the transition matrix, and draw the transition graph, for this new Markov chain.

1.3 *Gothenburg weather.* [Häg02, E2.3] Let us model the weather in Gothenburg by a Markov chain with state space $\Omega = \{\omega_1, \omega_2\}$ (with ω_1 = 'rain' and ω_2 = 'sunshine') and transition matrix

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}.$$

Suppose that the Markov chain starts on a rainy day, so that $\mu_0 = (1, 0)$.

(a) Prove by induction that

$$\mu_t = (\frac{1}{2}(1 + 2^{-t}), \frac{1}{2}(1 - 2^{-t})) \quad \text{for every } t \geq 1.$$

(b) What happens to μ_t in the limit as t tends to infinity?

Continues on the next page...

1.4 *Los Angeles weather.* [Häg02, E2.4]

(a) Let us model the weather in Los Angeles by a Markov chain with state space $\Omega = \{\omega_1, \omega_2\}$ (with $\omega_1 =$ 'rain' and $\omega_2 =$ 'sunshine') and transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}.$$

Suppose that the Markov chain starts with initial distribution $(\frac{1}{6}, \frac{5}{6})$. Show that $\mu_t = \mu_0$ for all $t = 0, 1, 2, \ldots$

(b) Can you find an initial distribution for the Markov chain in Exercise 1.3 for which we get similar behavior as in (a)? Compare this result to the one in Exercise 1.3 (b).

1.5 *Multi-step transition probabilities.* Let $(X_0, X_1, \ldots)$ be a finite Markov chain with transition matrix P. Show that

$$\mathbf{P}(X_{t+r} = y \mid X_t = x) = P^r(x, y)$$

for any states x and y, and all time indices $r, t \geq 0$.

References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications.* Cambridge University Press, 2002.