2.1 *Simulation of a discrete random variable.*[LPW08, Ex. B.2] Let U be a uniformly distributed random variable on $[0,1]$, and let X be the random variable $\phi(U)$, where ϕ is defined as in [LPW08, Eq. B.10]. Show that X takes on the value a_k with probability p_k.

2.2 *Ehrenfest urn.*[LPW08, Ex. 2.5] Let P be the transition matrix for the Ehrenfest chain described in [LPW08, Eq. 2.8]. Show that the binomial distribution with parameters n and $1/2$ is a stationary distribution for this chain.

2.3 *Partially observed Markov chain.* Let (X_t) be a finite Markov chain on Ω with initial distribution μ and transition matrix P. Define a random sequence $(Y_0, Y_1, \dots)$ by $Y_t = X_{rt}$, where r is a positive integer.

(a) Compute $\mathbf{P}(Y_1 = y \,|\, Y_0 = x)$.

(b) Show that (Y_t) is a Markov chain with initial distribution μ and transition matrix P^r.

2.4 *Sojourn time in a state.* Consider a finite Markov chain on Ω having a nonrandom initial state $x \in \Omega$. Assume that the transition matrix of the Markov chain satisfies $0 < P(x,x) < 1$. Denote the sojourn time at state x by $T = \min\{t \geq 1 : X_t \neq x\}$.

(a) Compute the probability $\mathbf{P}(X_1 = x, X_2 = x)$.

(b) Compute the probability $\mathbf{P}(X_1 = x, X_2 = x, X_3 \neq x)$.

(c) Compute the probability $\mathbf{P}(T = t)$ for all $t = 0, 1, 2, \dots$
Can you identify the distribution of T from this formula?

2.5 *Random walk on a connected graph.*[LPW08, Ex. 1.2] A graph G is *connected* when, for two vertices x and y of G, there exists a sequence of vertices $x_0, x_1, \dots, x_k$ such that $x_0 = x$, $x_k = y$, and $x_i \sim x_{i+1}$ for $0 \leq i \leq k-1$. Show that the simple random walk on G is irreducible if and only if G is connected.

References

[LPW08] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.