Exercise 4
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4.1 *Total variation distance.* Compute the total variation distance of μ and ν when:

(a) $\mu = (p, 1-p)$ and $\nu = (q, 1-q)$ are the Bernoulli distributions on $\{0, 1\}$ with parameters p and q, respectively.

(b) μ is the uniform distribution on $\Omega = \{1, 2, \dots, 19\}$ and $\nu = \delta_9$ is the distribution concentrated at state 9.

4.2 *Nonexpansive property.* [LPW08, Ex. 4.3]. Let P be the transition matrix of a Markov chain with state space Ω and let μ and ν be any two distributions on Ω.

(a) Prove that

$$||\mu P - \nu P||_{\mathrm{TV}} \le ||\mu - \nu||_{\mathrm{TV}},$$

(b) Assume that π is stationary distribution of P, and let $\mu_t = \mu_0 P^t$ for some initial distribution μ_0. Show with the help of a) that

$$||\mu_{t+1} - \pi||_{\mathrm{TV}} \le ||\mu_t - \pi||_{\mathrm{TV}},$$

that is, advancing the chain can only move it closer to stationarity.

4.3 *Random chess moves.* [Häg02, Ex. 4.3]

(a) Consider a chessboard with a lone white king making random moves, meaning that at each move, he picks one of the possible squares to move to, uniformly at random. Is the corresponding Markov chain irreducible and/or aperiodic?

(b) Same question, but with the king replaced by a bishop.

(c) Same question, but instead with a knight.

4.4 *Aperiodicity condition for irreducible Markov chains.* [Häg02, Ex. 4.2] Show that if a Markov chain is irreducible and has a state x such that $P(x, x) > 0$, then it is also aperiodic.

4.5 *Reducible Markov chain.* [Häg02, Ex. 5.2] Consider the reducible MC on $\Omega = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

(a) Show that both $\pi = (0.375, 0.625, 0, 0)$ and $\pi' = (0, 0, 0.5, 0.5)$ are stationary distributions for this Markov chain.

(b) Find an initial distribution μ_0 such that $||\mu_0 P^t - \pi||_{\mathrm{TV}} \to 0$ as $t \to \infty$.

(c) Find an initial distribution μ_0 such that $||\mu_0 P^t - \pi'||_{\mathrm{TV}} \to 0$ as $t \to \infty$.

(d) Can you say something about the convergence or divergence of $\mu_0 P^t$ when μ_0 is an arbitrary initial distribution on Ω ?

References

[Häg02] Olle Häggström. *Finite Markov chains and Algorithmic Applications*. Cambridge University Press, 2002.

[LPW08] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. American Mathematical Society, 2008.