University of Jyväskylä
Department of Mathematics and Statistics
MATS255 Markov Processes

**Exercise 5**
Mon 3 Dec 2012
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### Gibbs sampler for random hardcore configurations

Let $G = (V, E)$ be a graph with nodes $V = \{v_1, \dots, v_n\}$ and edges $E = \{e_1, \dots, e_\ell\}$. Denote by $\{0,1\}^V$ the set of functions from $V$ into $\{0,1\}$. An element $x \in \{0,1\}^V$ is called a *configuration*, and we say that a vertex $v$ is occupied if $x(v) = 1$ and vacant otherwise. An element $x \in \{0,1\}^V$ is called a feasible hardcore configuration on $G$ if no neighboring vertices of $G$ are occupied. We denote the set of feasible configurations by

$$\Omega = \{x \in \{0,1\}^V : x(v) + x(w) \le 1 \text{ for all } \{v, w\} \in E\}.$$

Construct a Markov chain $(X_t)$ on $\Omega$ recursively by letting $X_0$ be an arbitrary feasible configuration on $G$, and at each integer time $t+1$:

1. Pick a vertex $v \in V$ uniformly at random.
2. Toss a fair coin.
3. If the coin comes up heads, and all neighbors of $v$ take value 0 in $X_t$, then let $X_{t+1}(v) = 1$; otherwise let $X_{t+1}(v) = 0$.
4. For all nodes $w$ other than $v$, leave the value at $w$ unchanged, i.e., let $X_{t+1}(w) = X_t(w)$.

**5.1** Write down the transition matrix $P$ of the Markov chain $(X_t)$.

**5.2** Show that the uniform distribution on $\Omega$ is reversible for $P$. Can you find a simple proof of this fact using a simple structural property of $P$?

**5.3** Show that $P$ is aperiodic. (**Hint:** Show that $1 \in \mathcal{T}(x)$ for all $x$, where $\mathcal{T}(x)$ denotes the set of possible return times to configuration $x$ as defined in [LPW08, Sec 1.3].)

**5.4** Show that $P$ is irreducible. (**Hint:** Show first that the empty configuration can be reached from any feasible configuration in a finite number of steps.)

**5.5** Compute the transition matrix $Q$ of the Glauber dynamics for $\pi$ described in [LPW08, Sec 3.3.2], when $\pi$ is the uniform distribution on $\Omega$. In what way (if any) the matrix $Q$ differs from $P$?

## References

[LPW08] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.