University of Jyväskylä
Department of Mathematics and Statistics
MATS255 Markov Processes

Exercise 6
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6.1 *Ordered coupling.* Let μ and ν be probability distributions on $\Omega = \{0, 1, \dots, n\}$. Assume that (X, Y) is a coupling of μ and ν such that $\mathbf{P}(X \le Y) = 1$. Show that $\mathbf{P}(X > t) \;\le\; \mathbf{P}(Y > t)$ for all integers t.

6.2 *Metropolis chain with a general base chain.* [LPW08, Ex 3.1]. Let Ψ be an irreducible transition matrix on Ω, and let π be a probability distribution on Ω. Show that the transition matrix

$$P(x, y) = \begin{cases} \Psi(x, y) \left[\frac{\pi(y)\Psi(y,x)}{\pi(x)\Psi(x,y)} \wedge 1 \right] & \text{if } y \neq x, \\ 1 - \sum_{z \neq x} \Psi(x, z) \left[\frac{\pi(z)\Psi(z,x)}{\pi(x)\Psi(x,z)} \wedge 1 \right] & \text{if } y = x \end{cases}$$

defines a reversible Markov chain with stationary distribution π

6.3 *Glauber dynamics.* [LPW08, Ex 3.2]. Verify that the Glauber dynamics for π is a reversible Markov chain with stationary distribution π.

6.4 *Distance to stationarity is decreasing.* [LPW08, Ex. 4.4]. Let P be the transition matrix of a Markov chain with stationary distribution π. Prove that for any $t \ge 0$,

$$d(t + 1) \le d(t),$$

where $d(t)$ is defined by [LPW08, Eq 4.2].

6.5 *Distance to stationarity from a random initial state.* [LPW08, Ex. 4.1]. Prove that the distances $d(t)$ and $\bar{d}(t)$ defined in [LPW08, Sec 4.4] satisfy

$$d(t) = \sup_{\mu} ||\mu P^t - \pi||_{\mathrm{TV}},$$
$$\bar{d}(t) = \sup_{\mu,\nu} ||\mu P^t - \nu P^t||_{\mathrm{TV}},$$

where μ and ν vary over probability distributions on a finite set Ω.

References

[LPW08] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.