Exercise 1
Tue 20 Mar 2012
L. Leskelä

1.1 *Discrete random variables.* Let P be the uniform distribution in the finite set $\Omega = \{1, 2, \ldots, n\}$, that is,

$$P(A) = n^{-1}|A|, \quad A \subset \Omega,$$

where $|A|$ is the number of elements in A. Define the random variables U and V by $U(\omega) = \omega$ and $X = U^2$. Compute for U and X:

(a) probability mass function,

(b) distribution,

(c) expectation.

1.2 *Continuous random variables.* Let P the uniform distribution on $(0, 1)$, that is,

$$P(A) = \int_A dx$$

for all (Borel) measurable $A \subset (0, 1)$. Define the random variables U and Y by $U(\omega) = \omega$ and $Y = -(1/\lambda)\log U$, where $\lambda > 0$. Compute for U and X:

(a) cumulative distribution function,

(b) probability density function,

(c) distribution,

(d) expectation.

1.3 *Sum estimate and distribution.*

(a) Prove that for arbitrary events $A_1, A_2, \ldots$:

$$P(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} P(A_i).$$

(b) Let $X = (X_1, \ldots, X_d)$ be a random vector on the probability space $(\Omega, \mathcal{F}, P)$. Show that the distribution of X,

$$P_X(B) = P(X \in B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

is a probability measure on $\mathbb{R}^d$.

Continues on next page...

1.4 *Independent uniform random variables.* Let U_1 and U_2 be independent uniformly distributed random variables on $(0, 1)$, so that the distribution of both random variables is the uniform distribution on $(0, 1)$ as in Exercise 1.2. Compute the cumulative distribution function, probability density function, and the distribution for

(a) $U_1 U_2$,

(b) U_1/U_2.

1.5 *Mappings of independent random variables.* Consider a random sequence $(X_1, X_2, \dots)$, where $X_i : \Omega \to \mathbb{R}^{d_i}$. Prove that the following statements are equivalent:

- Random vectors $X_1, X_2, \dots$ are independent.
- Random variables $f_1(X_1), f_2(X_2), \dots$ are independent for all measurable $f_i : \mathbb{R}^{d_i} \to \mathbb{R}$.