2.1 *Characteristic functions of known distributions.* Compute the characteristic functions of the following random variables:

(a) $\xi \in \{0, 1\}$ which is Bernoulli distributed with parameter p.

(b) $B \in \{0, 1, \dots, n\}$ which is binomially distributed with parameters n and p.

(c) $M \in \mathbb{R}_+$ which is exponentially distributed and $EM = 1/\mu$ for some $\mu > 0$.

2.2 *Symmetric random variable.* A random variable X is *symmetric* if $-X$ and X share the same distribution. Show that X is symmetric if and only if ϕ_X is real-valued.

2.3 *Expectation of the product of independent random variables.* Let X and Y be independent random variables. Assume that $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^1(\Omega, \mathcal{F}, P)$.

(a) Show that $XY \in L^1(\Omega, \mathcal{F}, P)$.

(b) Show that

$$EXY = EX\,EY.$$

(c) Do these conclusions generalize to complex-valued random variables?

2.4 *Inversion formula for the distribution of a random integer.* Let Z be integer-valued random variable having the probability mass function $p_n = P(Z = n)$, $n \in \mathbb{Z}$. Show that

$$p_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iun} \phi_Z(u)\, du, \quad n \in \mathbb{Z}.$$

(Hint: If you want, you can first prove the formula $\int_0^{n\pi} \cos(t)\, dt = 0$, $n \in \mathbb{Z}_+$.)

2.5 *Characteristic function determines the distribution.* Consider random variables X and Y which have distributions μ_X and μ_Y, and cumulative distribution functions F_X and F_Y. Assume that the characteristic functions of X and Y satisfy $\phi_X = \phi_Y$.

(a) Explain why $F_X(b) - F_X(a) = F_Y(b) - F_Y(a)$ for all $a, b \in \mathbb{R} \setminus (D_X \cup D_Y)$, where D_X and D_Y designate the points of discontinuity for X and Y. (Hint: Lévy's inversion formula [Sottinen, L6.3.7].)

(b) Show that $F_X(b) = F_Y(b)$ for all $b \in \mathbb{R} \setminus (D_X \cup D_Y)$.

(c) Show that $F_X = F_Y$ (Hint: A cumulative distribution function has at most a countably many points of discontinuity.)

(d) Show that $\mu_X = \mu_Y$. (Hint: Dynkin's extension theorem [Sottinen, L2.3.8].)