**Exercise 3**
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L. Leskelä

**3.1** *Continuity of linear combinations.* Let $(X_n)$ and $(Y_n)$ be random sequences in $\mathbb{R}^d$ and $a, b \in \mathbb{R}$. Which of the following statements are true:

(a) $X_n \overset{\text{a.s.}}{\to} X$ and $Y_n \overset{\text{a.s.}}{\to} Y \implies aX_n + bY_n \overset{\text{a.s.}}{\to} aX + bY$.

(b) $X_n \overset{P}{\to} X$ and $Y_n \overset{P}{\to} Y \implies aX_n + bY_n \overset{P}{\to} aX + bY$.

(c) $X_n \overset{L^p}{\to} X$ and $Y_n \overset{L^p}{\to} Y \implies aX_n + bY_n \overset{L^p}{\to} aX + bY$.

**3.2** *Integrability of a positive random variable.* Let $X$ be a positive random variable with distribution $\mu$, so that the probability measure $\mu$ satisfies $\mu(\mathbb{R}_+) = 1$.

(a) Show that $EX < \infty$ if and only if $\sum_{n \ge 0} P(X > n) < \infty$.
(**Hint**: Write $P(X > n) = \int_{\mathbb{R}_+} 1(x > n)\, \mu(dx)$, where $1(x > n)$ is a convenient alternative way to write $1_{(n,\infty)}(x)$.)

(b) Can you interpret the sum $\sum_{n \ge 0} P(X > n)$ as the expectation of some random variable?

**3.3** *Growth speed of a random sequence.* Let $\xi, \xi_1, \xi_2, \dots$ be independent and identically distributed positive random variables. Denote $\xi_n = O(n^\beta)$ if there exists a number $c$ such that $\xi_n \le cn^\beta$ for all $n$ from some $n_0$ onwards.

(a) Show that for all $\alpha > 0$ and $c > 0$, the following are equivalent:

- $\xi_n \le cn^{1/\alpha}$ eventually (from some index onwards) almost surely.
- $E\xi^\alpha < \infty$.

(**Hint**: Exercise 3.2 and Borel–Cantelli.)

(b) Show that if $E\xi^\alpha < \infty$, then $\xi_n = O(n^{1/\alpha})$ almost surely.

(c) Show that if $E\xi^\alpha = \infty$, then $\limsup_{n\to\infty} n^{-1/\alpha}\xi_n = \infty$ almost surely.

(d) Based on the above findings, can you conclude something on the behavior of the record sequence $M_n = \max(\xi_1, \dots, \xi_n)$ for large $n$?

**3.4** *Almost sure convergence is also stochastic.* Let $(X_n)$ be a random sequence in $\mathbb{R}^d$. Prove the following theorem [Sottinen, Theorem 7.3.2(i)]: If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

**3.5** *Almost sure dominated convergence.* Show that Lebesgue's dominated convergence theorem [Sottinen, Lause 4.2.1(v)] is valid, when we only assume that $X_n \overset{\text{a.s.}}{\to} X$ and $|X_n| \le Y$ almost surely for all $n$, where $Y$ is an integrable random variable.